132305

ISSN 0304-9892

Jñānābha ज्ञानाभ

(10)

VOLUME 38

2008

Published by :

The Vijñāna Parishad of India

DAYANAND VEDIC POSTGRADUATE COLLEGE

(Bundelkhand University)

ORAI, U.P., INDIA

JÑĀNĀBHA

EDITORS

CONTENTS

INFORMATION FOR CONTRIBUTORS/SUBSCRIBERS

Jñānābha is published annually. It is an interdisciplinary journal devoted primarily to research articles in *all* areas of the mathematical, physical and biological sciences; it does,however, encourage original mathematical works which are motivated by and relevant to applications in the social, management, biomedical or engineering sciences. Papers intended for publication in this journal should be in typed form or offset-reproduced (*not dittoed*), A4 size double spaced with generous margins and they may be written in Hindi or English. **Manuscripts in duplicate (hard copy along with soft copy typed in MS word/Page Maker), mentioning 2000 Mathematics Subject Classification, Key Words and authors E-Mail addresses also on front page, if possible, may be submitted to *either* of the Editors.** *It is mandatory for every author in Jñānābha to be a member of the Vijñāna Parishad of India in good standing and to pay publication charges which will be based upon the number of printed pages.*

The submission of a paper implies the author's assurance that the paper has not been widely circulated, copyrighted, published or submitted for publication elsewhere.

Authors are advised to submit *only* neatly (and carefully) type-written and throughly checked manuscripts employing accepted conventions of references, notations, displays, etc.; *all typescripts* not in a format suitable for publication in this journal will be returned *unrefereed.*

25 reprints are provided free for each article; additional reprints, in multiples of 25 may be obtained at cost price, if the authors give prior intimation while returning proofs.

Effective with Volume 34 the price per volume is Rs. 200.00 (or U.S. $25.00). Individual members of the *Vijñāna Parishad of India* are entitled to free subscriptions to the current issues of this journal. Individual membership: Rs. 100.00 (or U.S. $20.00) per calender year; Life membership : Rs. 1000.00 (or U.S. $200.00). Back volumes are available at special price. (*Payment of dues by cheques must include appropriate bank charges.)*

[By a reciprocity agreement with the **American Mathematical Society,** an individual/ life member of the Parishad residing outside North American continent may join the Society by submitting an application for membership on the form that may be obtained from the office of the Society **(P.O. Box 6248, Providence, Rhode, Island 02940, U.S.A.)** and by paying the Society's current dues as a *reciprocity member* at a considerably reduced rate; the usual requirements that the applicant be endorsed by two members of the Society and that the candidate be elected by the Council are waived, but this reduction in dues to the Society does not apply to those individual/life members of the Parishad who reside, even on a temporary basis, in the North American area (i.e., U.S.A. and Canada).

The mathematical content of this journal is indexed in the Society's biweekly **Current Mathematical Publications** and is reviewed, among others, by **Mathematical Reviews** (U.S.A), **Zentralblatt für Mathematik** (Germany) and **Applied Mechanics Reviews** (U.S.A.).]

All communications regarding subsriptions, order for back volumes, membership of the *Vijñāna Parishad of India*, change of address, etc. and all books for review, should be addressed to:

The Secretary
Vijñāna Parishad of India
D.V. Postgraduat e College Orai 285001, U.P., India or
77DA, Sheeshmahal Apartment, Shalimar Gaon, New Delhi
E-Mail :rc_chandel@yahoo.com
Phone : 05162-251607

Jñānābha, Vol. 38, 2008

FINITE CAPACITY QUEUEING SYSTEM WITH QUEUE DEPENDENT SERVERS AND DISCOURAGEMENT

By

Madhu Jian and Pankaj Sharma

Department of Mathematics, Institute of Basic Science

Dr. B.R. Ambedkar University, Agra-282002, Uttar Pradesh, India

E-mail : madhujain@sancharnet.in and sharma_ibspankaj@yahoo.co.in

(Received : November 15, 2007)

ABSTRACT

An analytical study for optimal threshold policy for queueing system with state dependent heterogeneous servers and discouragement is presented. In this investigation the discouraging behavior of the customers has been considered due to which the customers may balk or renege. To reduce the discouraging behavior of the customers there is a provision of removable heterogeneous servers. The service rate of the servers are different and the number of servers in the system changes depending on the queue length. The first server starts service only when N customers are accumulated in the queue and once he starts serving, continues tc serve until the system becomes empty. The $j^{\text{th}}(j=2,3...,r)$ server turns on when there are N_{j-1} customers present in the system and are removed when the queue size ceases to less than N_{j-1}. By employing the recursive method we derive the steady state characteristics of the system such as queue size distribution, the average number of customers in the system, the average number of waiting customers, etc. Sensitivity analysis have been facilitated by taking numerical illustration.

2000 Mathematics Subject Classification : Primary 90B15; Secondary 90B50

Keywords and Phrases : Finite capacity, Queue dependent servers, Discouragement, Heterogeneous servers, Queue size, Threshold policy.

1. Introduction. In real life situations, it is common to use some extra servers to reduce the congestion by assuming that the number of servers changes according to the queue length. The decision makers often employ some heterogeneous removable servers in the system to reduce the discouraging behavior of customers and waiting time. Such situations can be encountered in many day-to-day congestion situations including banks, check-out counters, super market, cafeterias, petrol pumps, etc.

There are several queueing systems for which based on cost criteria, it is recommended that the number of servers should be increased one by one depending

upon the queue length. Garg and Singh (1993) investigated queue dependent servers queueing system to determine the optimal queue length at which the next server is provided in order to gain the maximum profit. Yamashiro (1996) considered a system where the number of servers changes depending on the queue length. A queueing system with queue dependent servers and finite capacity was considered by Wang and Tai (2000). Jain (2005) analysed finite capacity $M/M/r$ queueing system with queue dependent servers. Processor-shared service system with queue-dependent processors was also studied by Jain et al. (2005).

In classical N-policy queueing system, when there are N customers present in the system only then the server starts providing service to the customers. Many researchers have contributed to the studies on N-policy models in different framewaorks. Larsen and Agrawala(1983) investigated optimal policy to minimize the mean response time for $M/M/2$ queueing system. Medhi and Templeton (1992) analyzed the Poisson input queue under N-policy and with a general start up time. Kavusturucu and Gupta (1998) developed a methodology for analyzing finite buffer tandem manufacturing system with N-policy. Jau (2003) and Jain (2003) considered the operating characteristics for a general input queue and redundant arrival system, respectively under N-policy. Jain et al. (2004) analyzed N-policy for a machine repair system with spares and reneging. A two-threshold vacation policy for multi server queueing system was given by Tian and Zhey (2006).

The discouraging behavior of the customers has also been incorporated by several researchers while developing the queueing models of real life congestion problems. A prospective customer on arrival may join the queue or may balk depending on the number of customers present in the queue. Ankar and Gafarian (1963) analyzed some queueing problems with reneging. Blackburn (1972) and Gupta (1994) discussed different types of queues with balking and reneging. Jain and Vaidhya (1999) considered the multi server queue with discouragement and additional servers. Jain and Sharma (2002) investigated a multi server queue with additional server and discouragement.

To reduce the discouraging behavior of the customers, one of the important attributes to be considered by system organizers is to increase the number of servers while analyzing the congestion situations in different frameworks. In order to reduce the balking/reneging, the service providers can make the provision of the removable additional servers apart from some permanent servers from cost/space constraint view point. Abou and Shawky (1992) considered the additional servers in the single server Markovian over flow queue with balking and reneging. Jain (1998) analysed $M/M/m$ queue with discouragement and additional servers. Jain and Singh (2002) considered a $M/M/m$ queue with balking, reneging and additional servers. A multi server queueing model with discouragement and additional servers

was studied by Jain and Singh (2004) in order to facilitate a comparative study for multi servers queueing system with and without additional servers. Controllable multi server queue with balking was investigated by Jain and Sharma (2005). In this model they have incorporated an additional server which is added and removed at pre-specified threshold level of queue size to control the balking behavior of the customers.

In this paper we consider a finite capacity Markov queueing system with queue dependent heterogeneous servers and discouragement uder optimal threshold policy. The concept that the heterogeneous servers may be employed in the system one by one depending on the queue length can be helpful to reduce cost as well as the discouraging behavior of the customers in the system. The steady state queue size distribution by using recursive method is obtained which is further used to determine various system characteristis. The remaining part of the paper is organized as follows. In section 2, the model is described by stating requisite notations and assumptions. Queue size distribution and other system metrics have been derived in section 3 and 4, respectively. Some special cases are deduced by setting appropriate parameters, in Section 5. Sensitivity analysis is given in section 6. Section 7 concludes the paper and highlights the future scope of the model.

2. The Model . Consider a Markov queueing system with finite capacity and queue dependent heterogeneous servers. The concept of discouragement and heterogeneous removable servers under optimal control policy are taken into consideration. We assume that queue dependent $r(>1)$ servers offer services to the customers who arrive in Poisson fashion. By the queue dependent heterogeneous servers we mean to say that the servers turn on one by one depending upon the queue length according to a pre-specified rule and renders service with different rates, the service times taken by each server, are assumed to be exponential distributed. The arriving customers may balk with probability 1-b_j, $j(j=0,1,...,r)$ denotes the number of servers rendering service in the system. The customers may also renege from the queue after waiting for some time; according to exponential distribution with parameter α_j, $(j=0,1,2,...,r)$ here j indicates the number of servers present in the system. The customers are served according to first come first serve discipline and the capacity of the system including those in service is of size K.

The state dependent arrival rate of the customers are given as follows :

$$\lambda(n)=\begin{cases}\lambda b_1; & 0\le n\le N_1\\ \lambda b_j; & N_{j-1}\le n\le N_j, j=2,3,...,r-1\\ \lambda b_r; & N_{r-1}\le n\le K\end{cases} \qquad ...(1)$$

The effective service rates after incorporating the reneging concept, can be

given by :

$$\mu(n)=\begin{cases}\mu_0+(n-1)\alpha_0; & 1\le n\le N\\ \mu_1+(n-1)\alpha_1; & N\le n\le N_1\\ \sum_{i=1}^{j}\mu_i+(n-j)\alpha_j; & N_{j-1}<n\le N_j, j=2,3,\dots,r-1\\ \sum_{i=1}^{r}\mu_i+(n-r)\alpha_r; & N_{r-1}<n\le K\end{cases} \quad \dots(2)$$

The number of servers employed in the system depends upon the number of customers present in the system according to the following threshold policy:

- The first server turns on when $N(>1)$ cutomers are present in the system and turns off as soon as the system become empty.
- When there are more than N customers waiting in queue he provides service to the customers with the faster rate μ_1, otherwise serves the customers with rate μ_0.
- The j^{th} $(2,3,4,\dots,r)$ server provides service to the customers when there are more than N_{j-1} customers present in the system, and as soon as the queue length become less than N_{j-1} the j^{th} server is removed from the system.

Let $P(j,n)$ denotes the steady state probability that there are $n(n\geq1)$ customers present in the system and $j(j=1,2,\dots,r)$ heterogeneous servers are providing service to the customers. In our Model n denote the number of customers present in the system, $i(i=0,1,2)$ denotes the level of the service state and $j-1$ $(2,3,\dots,r)$ denotes the number of removable heterogeneous servers employed in the system.

Let $P(0,n)$ denote the steady state probability that there are 'n' customers in the system before start of the service $0\le n\le N-1$. $P(j,N_j(i))$ denotes the probability that there are N_j customers present in the system being served by the newly added server or previously existing server when I takes value 1 or 2 respectively, and there are j servers in the system. We denote the probability of N_j customers in the system by $P(j,N_j)=P(j,N_j(1))+P(j,N_j(2))$. Here $P(j,N_j(1))$ is the probability that the N_j^{th} customer in the system is being served by the j^{th} server while $P(j+1,N_j(2))$ denotes the probability that N_j^{th} customer is being served by the $(j+1)^{th}$ server, while $P(j+1,N_j(2))$ denotes the probability that N_j^{th} customer is being served by the $(j+1)^{th}$ server without having any reneging probability. We have also considered that when there are N_r customers waiting before the servers, then all the 'r' removable servers are available for service. The steady state equations for the

finite source multi server queueing system are given by

$$\lambda b_1 P(0,0) = \mu_0 P(1,1) \qquad \text{...(3)}$$

$$\lambda b_1 P(0,n) = \lambda b_1 P(0,n-2);\, 1 \le n \le N-1 \qquad \text{...(4)}$$

$$(\lambda b_1 + \mu_0) P(1,1) = (\mu_0 + \alpha_0) P(1,2) \qquad \text{...(5)}$$

$$\{\lambda b_1 + \mu_0 + (n-1)\alpha_0\} P(1,n) = \lambda b_1 P(1,n-1) + (\mu_0 + n\alpha_0) P(1,n+1);\, 2 \le n \le N \qquad \text{...(6)}$$

$$\{\lambda b_1 + \mu_0 + (N-1)\alpha_0\} P(1,N) = \lambda b_1 P(0,N-1) + \lambda b_1 P(1,N-1) + (\mu_1 + N\alpha_1) P(1,N+1); \text{...(7)}$$

$$\{\lambda b_1 + \mu_1 + (n-1)\alpha_1\} P(1,n) = \lambda b_1 P(1,n-1) + (\mu_1 + n\alpha_1) P(1,n+1);\, N+1 \le n \le N_1 - 2 \text{...(8)}$$

$$\{\lambda b_1 + \mu_1 + (N_1-2)\alpha_1\} P(1,N_1-1) = \lambda b_1 P(1,N_1-2) + \{\mu_1 + (N_1-1)\alpha_1\} P(1,N_1(1))$$
$$+ \mu_2 P(1,N_1(2)); \qquad \text{...(9)}$$

$$\{\lambda b_2 + \mu_1 + (N_1-1)\alpha_1\} P(1,N_1) = \lambda b_1 P(1,N_1-1) + \mu_2 P(2,N_1+1) \qquad \text{...(10)}$$

$$(\lambda b_2 + \mu_2) P(1,N_1(2)) = \{\mu_1 + (N_1-1)\alpha_2\} P(2,N_1+1) \qquad \text{...(11)}$$

$$\{\lambda b_j + \phi_j(n)\} P(j,n) = \lambda b_j P(n-1) + \phi_j(n+1) P(j,n+1);\, N_{j-1} + 1 < n \le N_j - 2,$$
$$j = 1,2,\ldots,(r-1) \qquad \text{...(12)}$$

$$\{\lambda b_j + \phi_j(N_j-1)\} P(j,N_j-1) = \lambda b_j P(j,N_j-2) + \phi_j(N_j) P(j,N_j(1))$$
$$+ \mu_{j+1} P(1,N_j(2));\, j = 1,2,\ldots,(r-1) \qquad \text{...(13)}$$

$$\{\lambda b_{j+1} + \phi_j(N_j)\} P(j,N_j(1)) = \lambda b_j P(j,N_j-1) + \mu_{j+1} P(j+1,N_j+1);\, j = 1,2,\ldots,(r-1) \text{...(14)}$$

$$(\lambda b_{j+1} + \mu_{j+1}) P(1,N_j(2)) = \{\phi_{j+1}(N_j+1) - \mu_{j+1}\} P(j+1,N_j+1);\, j = 1,2,\ldots,(r-1) \qquad \text{...(15)}$$

$$\{\lambda b_r + \phi_r(n)\} P(r,n) = \lambda b_r P(r,n-1) + \phi_r(n+1) P(r,n+1);\, N_{r-1} < n < K \qquad \text{...(16)}$$

$$\phi_r P(r,K) = \lambda b_r P(r,K-1)$$

3. Queue Size Distribution. In this section, we derive the mathematical expressions for the queue size distribution and average queue length for finite capacity model by employing recursive methed.

For brevity of notaion, we denote

$$\phi_j(n) = \sum_{i=1}^{j} \mu_i + (n-j)\alpha_j; \qquad N_{j-1} < n \le N_j,\, j = 2,3,\ldots,r-1$$

$$\phi_r(n) = \sum_{i=1}^{r} \mu_i + (n-r)\alpha_r; \qquad N_{r-1} < n \le K$$

$$r_n = \begin{cases} \dfrac{\lambda b_1}{\mu_0 + (n-1)\alpha_o}; & 1 \le n \le N \\ \dfrac{\lambda b_j}{\sum_{i=1}^{j} \mu_i + (n-j)\alpha_j}; & N_{j-1} < n \le N_j, N = N_0, j = 1,2,\ldots,r-1; \end{cases}$$

$$A = \sum_{i=0}^{N-1} \prod_{j=0}^{i} r_{N-j}, \rho_{N_{j-1}} = \frac{\lambda b_j}{\sum_{i=1}^{j-1} \mu_i + \left(N_{j-1} - \overline{j-1}\right)\alpha_j}, r^{(j-1)} = \frac{\lambda b_j}{\mu_j}, R^{(j-1)} = \frac{\lambda b_{j-1}}{\mu_j}, j = 2,3,\ldots,r-1.$$

The steady state probability $P(1,n)$ is obtained by solving equations (3) to (8) as :

$$P(1,n) = \begin{cases} \sum_{i=0}^{n-1} \prod_{j=0}^{i} r_{n-j} P(0,0); & 1 \le n \le N \\ A \prod_{i=0}^{n-N} r_{N+i} P(0,0); & N \le n \le N_1 \end{cases} \qquad \ldots(18)$$

For $j = 2,3,\ldots,r-1$, we obtain probabilities using equations (9)-(15) as

$$P(j,n) = \frac{A \prod_{K=1}^{j-1} \left[\rho_{N_K} r^{(K)} \left(r^{(K)} + 1\right) R^{(K)}\right] \prod_{i=N+i}^{n-N} r_{N+i} P(0,0)}{\prod_{K=1}^{j-1} r_{N_{K+1}} \prod_{K=1}^{j-1} \left\{R^{(K)} \rho_{N_K} \left(r^{(K)} + 1\right) + r^{(K)} \left(r_{N_K} r^{(K)} + R^{(K)}\right)\right\}};$$

$$N_{j-1} < n \le N_j, j = 2,3,\ldots,r-1 \qquad \ldots(19)$$

Also solving equations (16) and (17), we get

$$P(r,n) = \frac{A \prod_{K=1}^{r-1} \left[\rho_{N_K} r^{(k)} \left(r^{(K)} + 1\right) R^{(K)}\right] \prod_{i=N+i}^{n-N} r_{N+i} P(0,0)}{\prod_{K=1}^{r-1} r_{N_{K+1}} \prod_{K=1}^{r-1} \left\{R^{(k)} \rho_{N_K} \left(r^{(K)} + 1\right) + r^{(k)} \left(r_{N_K} r^{(K)} + R^{(K)}\right)\right\}}; N_{r-1} < n \le K \qquad \ldots(20)$$

$P(0,0)$ can be obtained by using normalizing condition

$$\sum_{n=0}^{N-1} P(0,n) + \sum_{n=1}^{N_1} P(1,n) + \sum_{j=2}^{r-1} \sum_{n=N_{j-1}+1}^{N_j} P(j,n) + \sum_{n=N_{r-1}+1}^{K} P(r,n) = 1.$$

At the threshold level the probabilities are obtained as :

$$P(j,N_j(1)) = \frac{r_{N_j} R^{(j)}\left\{\rho_{N_j}\left(r^{(j)}+1\right)+r^{(j)}\right\}}{\left\{R^{(j)}\rho_{N_j}\left(r^{(j)}+1\right)+r^{(j)}\left(r_{N_j}r^{(j)}+R^{(j)}\right)\right\}} P(j,N_{j-1}); j=1,2,\ldots,r-1 \quad \ldots(21)$$

and

$$P(j,N_j(2)) = \frac{r_{N_j} R^{(j)}\left\{r^{(j)}\right\}^2}{\left\{R^{(j)}\rho_{N_j}\left(r^{(j)}+1\right)+r^{(j)}\left(r_{N_j}r^{(j)}+R^{(j)}\right)\right\}} P(j,N_{j-1}); j=1,2,\ldots,r-1 \quad \ldots(22)$$

Also

$P(j,N_j) = P(j,N_j(1)) + P(j,N_j(2)), j=1,2,\ldots,r-1.$

4. Performance Metrics. This section is devoted to some performance characteristics by using steady state queue size distribution as follows:
Pr (1 server is rendering service in the system)$=P(1)=Pr\ ob\{1<n\leq N_1\}$

$$= \sum_{n=1}^{N}\left(\sum_{i=0}^{n-1}\prod_{j=0}^{i} r_{n-j}P(0,0)\right) + \sum_{n=N+1}^{N_1} A\prod_{i=0}^{n-N} r_{N+1}P(0,0) \quad \ldots(23)$$

Pr (j servers are rendering service in the system)$=P(j)=Prob.\{N_{j-1}\leq n < N_j\}$

$$= \frac{A\prod_{K=1}^{j-1}\left[\rho_{N_K} r^{(K)}\left(r^{(K)}+1\right)R^{(K)}\right]\sum_{n=N_{j-1}}^{N_j-1}\prod_{i=N+i}^{n-N} r_{N+i}P(0,0)}{\prod_{K=1}^{j-1} r_{N_K+1}\prod_{K=1}^{j-1}\left\{R^{(K)}\rho_{N_K}\left(r^{(K)}+1\right)+r^{(K)}\left(r_{N_K}r^{(K)}+R^{(K)}\right)\right\}}; N_{j-1}<n\leq N_j, j=2,3,\ldots,r-1 \quad (24)$$

Pr (all r servers are rendering service in the system)$=P(r)=Prob\{N_{r-1}\leq n\leq K\}$

$$= \frac{A\prod_{K=1}^{j-1}\left[\rho_{N_K} r^{(K)}\left(r^{(K)}+1\right)R^{(K)}\right]\sum_{n=N_{j-1}}^{N_j-1}\prod_{i=N+i}^{n-N} r_{N+i}P(0,0)}{\prod_{K=1}^{j-1} r_{N_K+1}\prod_{K=1}^{j-1}\left\{R^{(K)}\rho_{N_K}\left(r^{(K)}+1\right)+r^{(K)}\left(r_{N_K}r^{(K)}+R^{(K)}\right)\right\}}; N_{j-1}<n\leq N_j, j=2,3,\ldots,r-1 \quad (25)$$

The average number of customers in the system having r heterogeneous queue dependent removable servers which are employed at thresholds $\underline{N}=(N,N_1,N_2,\ldots,N_{r-1})$ is obtained using:

$$L(r:N) = \sum_{n=0}^{K} nP_n$$

$$= \left[\sum_{n=0}^{N} \left(\sum_{i=0}^{n-1} \prod_{j=0}^{i} r_{n-j} \right) + A \sum_{n=N+1}^{N_1-1} \left(\prod_{i=0}^{n-N} r_{N+i} \right) \right.$$

$$+ A \sum_{j=2}^{r-1} \sum_{n=N_{j-1}+1}^{N_j} \frac{\prod_{K=1}^{j-1} \left[\rho_{N_K} r^{(K)} \left(r^{(K)}+1 \right) R^{(K)} \right] \prod_{i=N+i}^{n-N} r_{N+i}}{\prod_{K=1}^{j-1} r_{N_K+1} \prod_{K=1}^{j-1} \left\{ R^{(K)} \rho_{N_K} \left(r^{(K)}+1 \right) + r^{(K)} \left(r_{N_K} r^{(K)} + R^{(K)} \right) \right\}}$$

$$\left. + A \sum_{n=N_{r-1}}^{K} \frac{\prod_{K=1}^{r-1} \left[\rho_{N_K} r^{(K)} \left(r^{(K)}+1 \right) R^{(K)} \right] \prod_{i=N+i}^{n-N} r_{N+i}}{\prod_{K=1}^{r-1} r_{N_K+1} \prod_{K=1}^{r-1} \left\{ R^{(K)} \rho_{N_K} \left(r^{(K)}+1 \right) + r^{(K)} \left(r_{N_K} r^{(K)} + R^{(K)} \right) \right\}} \right] P(0,0) \qquad \text{...(26)}$$

The throughout of the system is given by

$$\tau = \sum_{n=1}^{K} \mu_n P_n \,. \qquad \text{...(27)}$$

5. Special Cases. In this section, we discuss some special cases that can be deduced from analytical results derive in the previous sections.

Case I: *M/M/R* Finite Capacity Queueing System With Queue Dependent Servers. In this case we set $b_j=1$ and $\alpha_j = 0(j = 0,1,2,...,r)$, and avoid the first threshold level i.e. we set $N=1$ then our results tally with those of Jain's (2005) model for *M/M/r* queue with heterogeneous queue dependent servers. In particular, we come across the following special situations:

i. If we set $r=3$, then our results coincide with the results obtained by Wang and Tai (2000).

ii. When the system capacity is infinite i.e. ($K\rightarrow\infty$), then the results correspond to the finite capacity queueing system with three queue dependent heterogeneous servers model which was discussed by Wang and Tai (2000).

iii. We can obtain the results for model of queue dependent homogeneous servers by setting $\mu_0 = \mu_1 = ... = \mu_r$ for this model.

Case II : When $N=1$, $N_1=2$, $N_3=3,...,N_r=r$, $b_j=1$ and $\alpha_j=0(j=0,1,2,...,r)$, then our model converts to *M/M/r* queueing model with heterogeneous server.

Case III: When $N=1$, $N_1=2$, $N_3=3,....,N_r=r$, $b_j=1$, $\alpha_j=0(j=0,1,2,...,r)$, $\mu_j=\mu$ then model provides results for classical *M/M/r* queueing model with homogeneous

server.

5. Sensitivity Analysis. In this section, the sensitivity analysis is performed to examine the effect of various system descriptors on the average queue length and throughout of the system. A computer program is developed by using the mathematical software *MATLAB* on Pentium IV. Various performance indices are computed and summarized in tables 1 to 3 by setting the default parameters as $N=1$, $K=50$, $r=3$, $b_1=0.4, \mu_0=1.5, \mu=1.5\,\mu_0, \alpha_0=0.2, \lambda=3$. Some more results are also obtained and displayed in figures 1-4 using the default parameters as $N=1$, $K=50$, $r=3$, $b_1=1$, $\mu_0=1.5, \mu=1.5\,\mu_0, \alpha_0=0.2, \lambda=1.5$. The following three policies having different threshold parameters are taken into consideration:

Policy 1: In this policy, the homogeneous servers turn on one by one with the arrival of each customers i.e. $\mu_1=\mu, N_j=j+1$.

Policy 2: In this case, the heterogeneous servers activate one by one with the arrival of customers i.e. $N_j=j+1$ also we set $\mu_j=1+0.1(j-1)\mu_0$.

Policy 3: For this policy, we consider the heterogeneous removable three servers who starts the service according to rule $N_j=3j$ and we chosen $\mu_j=1+0.1(j-1)\mu_0$.

Table 1 shows the effect of N, and arrival rate (λ). Both, average queue length (L) and through put (τ) of the system increase with the increase in λ and b_1. Tables 2 and 3 depict the effect of (N, μ_0) $\alpha\upsilon\delta$ (N, α) on the average queue length (L) and through put (τ). We note that the average queue length (L) and through put (τ) decrease as μ_0 increase but both L and τ increases as b_1 increases. Effect of N and α_0 are demonstrated in table 3. When we increase the α_0, L decreases and τ increases. but effect is not much significant.

From figures 1 and 2, we see that the average queue length (L) increases (decreases) with arrival rate λ (service rate μ) for the different policies, which is what we expect in the real life situations. Also it is noted from figures 3 and 4, that the average queue length (L) decreases (increases) with the increase in reneging parameter α_0 (joining parameter b_1) for the different policies. If homogeneous servers are taken into consideration i.e. for policy 1, the average queue length is higher in comparison to the policies 2 and 3, where heterogeneous servers are employed. The queue length is slightly higher in policy 3 where heterogeneous servers turn on with the additional workload of 3 customers, in comparison to policy 2 where heterogeneous servers turn on with the addition of one customer; this pattern matches with physical situations.

N	λ	L			τ		
		b_1=.4	b_1=.6	b_1=.8	b_1=.4	b_1=.6	b_1=.8
5	5	2.8551	4.4845	7.3264	6.5472	11.8068	16.3062
	8	4.9483	10.5025	18.1395	12.7928	19.4009	25.6156
	11	8.8264	19.1339	30.0820	17.8758	26.4102	35.1656
	14	14.2107	28.1095	40.2298	22.4691	33.5877	43.2838
	17	20.1303	36.5396	45.2481	27.2065	40.3317	47.2985
	20	26.1204	42.4439	47.2093	31.9965	45.0551	48.8675

Table 1: Effect of N and λ on Performance Measures

N	μ_0	L			τ		
		b_1=.4	b_1=.6	b_1=.8	b_1=.4	b_1=.6	b_1=.8
5	0.5	4.2780	6.6925	9.4514	5.2456	7.6043	9.7481
	1	2.5847	3.6305	5.2459	3.7634	6.9433	9.9072
	1.5	2.2325	2.6505	3.3785	3.0157	5.5534	8.6544
	2	2.1231	2.3311	2.6908	2.6952	4.7315	7.3479
	2.5	2.0762	2.1986	2.4039	2.5313	4.2772	6.4814
	3	2.0519	2.1322	2.2632	2.4354	4.0067	5.9306

Table 2: Effect of N and μ_0 on Performance Measures

N	α_0	L			τ		
		b_1=.4	b_1=.6	b_1=.8	b_1=.4	b_1=.6	b_1=.8
5	0.1	2.2632	2.7384	3.6382	2.9653	5.4715	8.5479
	0.2	2.2325	2.6505	3.3785	3.0157	5.5534	8.6544
	0.3	2.2065	2.5847	3.2166	3.0672	5.6451	8.7912
	0.4	2.1837	2.5318	3.1007	3.1194	5.7435	8.9474
	0.5	2.1633	2.4873	3.0113	3.1721	5.8472	9.1183
	0.6	2.1447	2.4488	2.9390	3.2252	5.9553	9.3014

Table 3: Effect of N and α_0 on Performance Measures

Over all, we conclude that the average queue length (L) reduces (increases) with the increase in N, μ_0 and α_0 (λ and b_1). The through put (τ) shows the increasing trends with the parameters N, λ, b_1, μ_0 and α_0 which is quite obvious. By incorporation of additional removable servers, there is remarkable decrement in the queue length which promotes the provision of a pool of additional servers along with permanent servers.

6. Discussion. In this paper, we have studied a finite capacity multi server queueing system with queue dependent removable servers and discouragement under N-policy. The assumption of discouraging behavior of customres makes our model more versatile as it deals with more realistic congestion situations. To reduce the cost, the decision makers are suggested to employ heterogeneous serves that may further be removed after pre specified threshold level. The cost relationship established is helpful to determine the optimal queue level to introduce the removable servers in order to gain the maximum net profit.

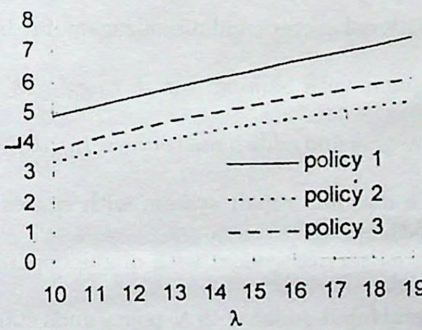


Fig 1: Average number of customers (L) vs. arrival rate (λ) for different policies.

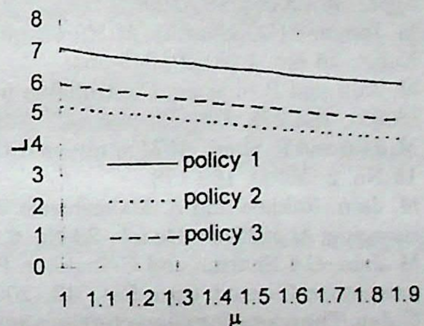


Fig 2: Average number of customers (L) vs. service rate (μ_0) for different policies

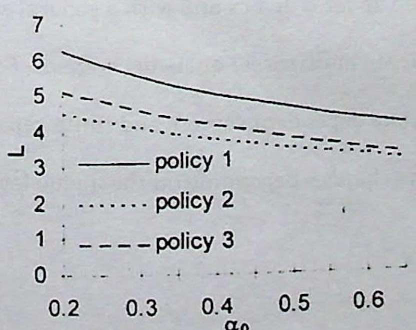


Fig 3: Average number of customers (L) vs. α_0 for different policies.

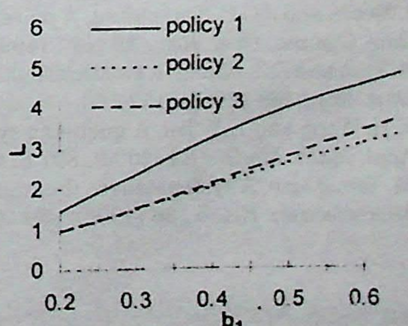


Fig 4: Average number of customers (L) vs. b_1 for different policies.

REFERENCES

[1] M.O. Abou-El-Ata and A.I. Shawky; The single server Markovian overflow queue with balking, reneging and an additional server for longer queues, *Microelectron. Reliab.*, **32** (1992), 1389-1394.

[2] C.J. Anker and A.V. Gafarian, Queueing with reneging and multiple heterogeneous servers, *Naval Res. Log., Quart,* **10** No. 1 (1963), 125-149.

[3] J.D. Blackburn, Optimal control of single server queue with balking and reneging, *Manage Sci.*, **19** No.8 (1972), 297-379.

[4] R.L. Garg and P. Singh, Queue dependent servers queueing system, *Microelectron. Reliab.* **33** (1993), 2289-2295.

[5] S.M. Gupta, Interrelationship between queueing models with balking and reneging and machine repair problem with warm spares. *Microelectron. Reliab.*, **34** No. 2, (1994), 201-209.

[6] M. Jain and P. Singh, A multi server queueing model with discouragement and additional servers, Recent Advances in Operations Research, Information Technology and Industry, (M. Jain and G.C. Sharma (Eds)), *S.R. Scientific Pub.*, Agra India (2004).

[7] M. Jain, *M/M/m* queue with discouragment and additional servers, *Journal of G.S.R.*, **36** No. 1-2, (1998), 30-42.

[8] M. Jain and V. Vaidhya, *M/M/m/K* queue with discouragement and additional servers, *Opsearch* **36** No.1 (1999), 73-80.

[9] M. Jain, N-policy for redundant repairable system with additional repairmen, *Opsearch,* **40** No. 2, (2003), 97-114.

[10] M. Jain, Finite capacity *M/M/r* queueing system with queue dependent servers, *Comput. Math. Appl.*, **50** (2005), 187-199.

[11] M. Jain and G.C. Sharma, *M/M/m/K* queue with additional servers and discouragement, *Int. J. Engg.*, **15** No. 4 (2002), 349-354.

[12] M. Jain and P. Sharma, Controllable multi server queue with balking, *Int. J. Engg.*, **18** No.3 (2005), 263-271.

[13] M. Jain and P. Singh, *M/M/m* queue with balking, reneging and additional servers, *Int. J. Engg.*, **15** No. 2 (2002), 169-178.

[14] M. Jain, Rakhee and S. Maheshwari, *N*-policy for a machine repair system with spares and reneging, *Appl.Math. Model.*, **28** No. 6 (2004), 513-531.

[15] M. Jain, G.C.Sharma and C.Shakhar, Processor shared service system with queue dependent processors, *Comput. Oper. Res.*, **32**, (2005), 629-641.

[16] C. Jau, The operating characteristics analysis on general input queue with *N*-policy anda start up times, *Math. Oper. Res.*, **57** No.2 (2003), 235-254.

[17] A. Kavusturucu and S.M. Gupta, A methodology for analyzing finite buffer tandem manufacturing systems witn *N*-policy, *Comput. Indust. Engg.*, **34** *issue* 4, (1998), 837-843.

[18] R.L. Larcen and A.K. Agrawala, Control of a heterogeneous two server exponential queueing system, *IEEE Trans. Soft. Engg.*, **9** (1983), 522-529.

[19] J. Medhi and J.G.C. Templeton; A poisson input queue under *N*-policy and with a general setup time, *Comput. Oper. Res.*, **19** No.1 (1992) 35-41.

[20] N. Tian and Z.G. Zhang, A two threshold vacation policy in multi server queueing systems, *Eur. J. Oper. Res.*, **168** (2006), 153-163.

[21] K.H. Wang and K.Y. Tai, A queueing system with queue dependent servers and finite capacity. *Appl. Math. Model.*, **24** (2000), 807-814.

[22] M. Yamashiro, A system where the number of servers changes depending on the queue length, *Microelectron. Relab.*, **36** (1996), 389-391.

Jñānābha, Vol. 38, 2008

SOME GENERATING FUNCTIONS OF CERTAIN POLYNOMIALS USING LIE ALGEBRAIC METHODS

By
M.B. El-Khazendar
Department of Mathematics
Alazhar University-Gaza, P.O. Box 1277, Gaza, Plestine
e-mail :emarawan@hotmail.com

(Received : November 15, 2007)

ABSTRACT

In ths present paper a group theoretic method is applied to the differential eqation whose solution is ${}_2F_1(-n,\alpha;v;f(x))$. Then we study in details the case $f(x)=e^x$. We consider six-parameter Lie group for this hypergeometric function, which does not seem to appear earlier. By means of this group theoretic method some new generating functions are obtained from which several new special generating functions can be easily derived.

2000 Mathematics Subject Classification : 22E30, 22E80, 33C05
Keywords and Phrases : Hypergeometric function, Lie group, generating functions

1. Introduction. The hypergeometric function of one variable is introduced by Gauss in (1822) and defined by

$$ {}_2F_1(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}=1+\frac{a.b}{1.c}z+\frac{a(a+1).b(b+1)z^2}{1.2.c.(c+1)}+\dots \tag{1.1}$$

where $(a)_n$ denote the Pochhammer symbol defined by

$$(a)_n=\begin{cases}1 & \text{if } n=0\\ a(a+1)\dots(a+n-1) & \text{if } n=1,2,3\dots\end{cases} \tag{1.2}$$

The hypergeometric polynomial (1.1) is a solution of the following equation :

$$\left[\frac{f^2(x)(1-f(x))}{[f'(x)]^2}\right]\frac{d^2y}{dx^2}+\left[\frac{f^2(x)(f''(x))}{[f'(x)]^3}(f(x)-1)-\frac{f(x)}{f'(x)}(-n+\alpha+1)f(x)-v\right]\frac{dy}{dx}$$
$$+n\alpha f(x)y=0 \tag{1.3}$$

Several generating relations for hypergeometric polynomials have been derived by different methods e.g. classical, theory of Lie-group etc. In a recent paper A.K. Chongdar [1] has derived some generating functions for the said polynomials by Lie algebraic method. See also [2] and [3].

Now, we can apply the group theoretic method on (1.3) by replacing

$\frac{d}{dx}$ by $\frac{\partial}{\partial x}$, $n = \frac{f(y)}{f'(y)}\frac{\partial}{\partial y}$, $\alpha = \frac{f(z)}{f'(z)}\frac{\partial}{\partial z}$ and $y = u(f(x), f(y), f(z))$ in (1.2).

We get the following partial differential equation :

$$\left(\frac{f^2(x)(1-f(x))}{(f'(x))^2}\right)\frac{\partial^2 u}{\partial x^2} + \frac{f^2(x)}{f'(x)}\cdot\frac{f(y)}{f'(y)}\frac{\partial^2 u}{\partial y \partial x} - \frac{f^2(x)}{f'(x)}\frac{f(z)}{f'(z)}\frac{\partial^2 u}{\partial z \partial x} +$$

$$f(x)\frac{f(y)}{f''(y)}\frac{f(z)}{f'(z)}\frac{\partial^2 u}{\partial y \partial z} + \left[\frac{f(x)}{f'(x)}\upsilon - \frac{f^2(x)}{f'(x)} + \frac{f^2(x)f''(x)(f(x)-1)}{(f'(x))^3}\right]\frac{\partial u}{\partial x} = 0 \tag{1.4}$$

Thus $u_1(f(x), f(y), f(z)) = {}_2F_1(-n, \alpha, \alpha; \upsilon(x))[f(y)]^n[f(z)]^\alpha$ is a solution of the differential equation (1.3) since ${}_2F_1(-n, \alpha, \upsilon; f(x))$ is a solution of (1.3) we now defined the infinitesimal operator $A_i (i = 1, 2, \ldots, 6)$

$$A_i = A_{ij}^{(1)}\frac{\partial}{\partial x} + A_{ij}^{(2)}\frac{\partial}{\partial y} + A_{ij}^{(3)}\frac{\partial}{\partial z} + A_{ij}^{(0)}; (j = 1, 2, \ldots, 6)$$

as follows :

$$\left.\begin{aligned}
A_1 &= \frac{f(y)}{f'(y)}\frac{\partial}{\partial y} \\
A_2 &= \frac{f(z)}{f'(z)}\frac{\partial}{\partial z} \\
A_3 &= \frac{f(x)}{f(y)f'(x)}\frac{\partial}{\partial x} - \frac{1}{f'(y)}\frac{\partial}{\partial y} \\
A_4 &= \frac{f(x)(1-f(x))f(y)}{f(x)}\frac{\partial}{\partial x} + \frac{[f(y)]^2}{f(y)}\frac{\partial}{\partial y} - \frac{f(x)f(y)f(z)}{f(z)}\frac{\partial}{\partial z} - \upsilon f(y) \\
A_5 &= \frac{f(x)(1-f(x))}{f(z)f'(x)}\frac{\partial}{\partial x} + \frac{f(x)f(y)}{f(z)f'(y)}\frac{\partial}{\partial y} - \frac{1}{f'(z)}\frac{\partial}{\partial z} + \upsilon f(z)^{-1} \\
A_6 &= \frac{f(x)f(z)}{f'(x)}\frac{\partial}{\partial x} + \frac{[f(z)]^2}{f'(z)}\frac{\partial}{\partial z}
\end{aligned}\right\} \tag{1.5}$$

2. Application. Now, we study in details the case when $f(x) = e^x$, then the differential equation (1.3) can be rewritten as

$$\left(1 - e^x\right)\frac{d^2 w}{dx^2} + \left[(n - \alpha)e^x + (\upsilon - 1)\right]\frac{dw}{dx} + n\alpha e^x w = 0, \tag{2.1}$$

where $W = {}_2F_1\left(-n, \alpha; \upsilon; e^x\right)$

Let $n = \frac{\partial}{\partial y}, \alpha = \frac{\partial}{\partial z}, \frac{d}{dx} = \frac{\partial}{\partial x}$ and $W = u\left(e^x, e^y, e^z\right)$,

Then from (1.3), we get

$$\left(1-e^{x}\right)\frac{\partial^2 u}{\partial x^2}+e^{x}\frac{\partial^2 u}{\partial y\partial x}-e^{x}\frac{\partial^2 u}{\partial z\partial x}+e^{x}\frac{\partial^2 u}{\partial y\partial z}+(v-1)\frac{\partial u}{\partial x}=0 \tag{2.2}$$

which has the solution

$$u\left(e^{x},e^{y},e^{z}\right)={}_2F_1\left(-n,\alpha;v;e^{x}\right)e^{ny}e^{\alpha z}. \tag{2.3}$$

So we define

$$\left.\begin{aligned}
A_1 &= \partial/\partial y\\
A_2 &= \partial/\partial z\\
A_3 &= e^{-y}\left(\partial/\partial x-\partial/\partial y\right)\\
A_4 &= \left(1-e^{x}\right)e^{y}\,\partial/\partial x+e^{y}\,\partial/\partial y-e^{x+y}\,\partial/\partial z-ve^{y}\\
A_5 &= e^{-z}\left(1-e^{x}\right)\partial/\partial x+e^{x-z}\,\partial/\partial y-e^{-z}\,\partial/\partial z+ve^{-z}\\
A_6 &= e^{z}\left(\partial/\partial x+\partial/\partial z\right)
\end{aligned}\right\} \tag{2.4}$$

such that

$$\left.\begin{aligned}
A_1\left[{}_2F_1\left(-n,\alpha;v;e^{x}\right)e^{ny+\alpha z}\right] &= n\ {}_2F_1\left(-n,\alpha;v;e^{x}\right)e^{ny+\alpha z}\\
A_2\left[{}_2F_1\left(-n,\alpha;v;e^{x}\right)e^{ny+\alpha z}\right] &= \alpha\ {}_2F_1\left(-n,\alpha;v;e^{x}\right)e^{ny+\alpha z}\\
A_3\left[{}_2F_1\left(-n,\alpha;v;e^{x}\right)e^{ny+\alpha z}\right] &= -n\ {}_2F_1\left(-(n-1),\alpha;v;e^{x}\right)e^{(n-1)+\alpha z}\\
A_4\left[{}_2F_1\left(-n,\alpha;v;e^{x}\right)e^{ny+\alpha z}\right] &= (v+n)\ {}_2F_1\left(-(n+1),\alpha;v;e^{x}\right)e^{ny+\alpha z}\\
A_5\left[{}_2F_1\left(-n,\alpha;v;e^{x}\right)e^{ny+\alpha z}\right] &= (v-\alpha)\ {}_2F_1\left(-n,\alpha-1;v;e^{x}\right)e^{ny+\alpha-1}\\
A_6\left[{}_2F_1\left(-n,\alpha;v;e^{x}\right)e^{ny+\alpha z}\right] &= \alpha\ {}_2F_1\left(-n,\alpha+1;v;e^{x}\right)e^{ny+\alpha+1}
\end{aligned}\right\}. \tag{2.5}$$

Now we have the following commutate or relations. Using the relation $[A,B]u=(AB-BA)u$, we get

$$\left.\begin{array}{llll}
[A_1,A_2]=0 & ;\ [A_2,A_3]=0 & ;\ [A_3,A_4]=-2A_1-V & ;\ [A_4,A_5]=0\\
[A_1,A_3]=-A_3 & ;\ [A_2,A_4]=0 & ;\ [A_3,A_5]=0 & ;\ [A_4,A_6]=0\\
[A_1,A_4]=A_4 & ;\ [A_2,A_5]=-A_5 & ;\ [A_3,A_6]=0 & ;\\
[A_1,A_5]=0 & ;\ [A_2,A_6]=A_6 & ;\ [A_5,A_6]=-2A_2+v & \\
[A_1,A_6]=0 & ; & &
\end{array}\right\} \tag{2.6}$$

Hence we get the following theorem :

Theorem. The set $(I,A_i\ (i=1,2,\ldots,6))$ when I stands for the indentity operator, generates a Lie-algebra L and each of the sets

$\{I,A_i(i=1,2,3,4)\}$ and $\{I,A_j(j=1,2,5,6)\}$

forms a subalgebra of L.

It can be easily shown that the partial differential operator L given by :

$$L=\left(1-e^{x}\right)\frac{\partial^2}{\partial x^2}+e^{x}\frac{\partial^2}{\partial y\partial x}-e^{x}\frac{\partial^2}{\partial z\partial x}+e^{x}\frac{\partial^2}{\partial y\partial z}+(\upsilon-1)\frac{\partial}{\partial x} \tag{2.7}$$

can be related to each A_i in the following two different ways, i.e.

$$\left.\begin{aligned} L &= A_4A_3 + A_1(A_1+\upsilon-1) \quad \text{and} \\ L &= A_6A_5-(A_2-1)(\upsilon-A_2) \end{aligned}\right\}. \tag{2.8}$$

From which it follows that L commutes with each of the A_i

$$\text{i.e. } [A_i, L]=0\;(i=1,2,\dots,6). \tag{2.9}$$

. The extended form of the groups generated by $(A_i\,(i=1,2,\dots,6))$ given by;

$$e^{a_1A_1}u\left(e^{x},e^{y},e^{z}\right)=u(x,a_1+y,z) \tag{2.10}$$

$$e^{a_2A_2}u\left(e^{x},e^{y},e^{y}\right)=u(x,y,a_2+z) \tag{2.11}$$

$$e^{a_3A_3}u\left(e^{x},e^{y},e^{y}\right)=u\left(y+x-\text{In}\left(e^{y}-a_3\right),\text{In}\left(e^{3}-a_3\right),z\right) \tag{2.12}$$

$$e^{a_4A_4}u\left(e^{x},e^{y},e^{y}\right)=-\upsilon\,\text{In}\left(1-a_4e^{y}\right)$$

$$u\left(\left(x-\text{In}\left(1-a_4e^{y}\left(1-e^{x}\right)\right)\right),y-\text{In}\left(1-\alpha_4e^{y}\right)\right),z+\text{In}\left(\frac{1-a_4e^{y}}{1-a_4e^{y}\left(1-e^{x}\right)}\right)\right) \tag{2.13}$$

$$e^{a_5A_5}u\left(e^{x},e^{y},e^{y}\right)=-\upsilon\,\text{In}\left(\frac{e^{z}}{e^{z}-a_5}\right)u\left(x+y-\text{In}\left(e^{z}-a_5\left(1-e^{x}\right)\right)\right)$$

$$e^{y}+\text{In}\left(\frac{e^{z}-a_5\left(1-e^{x}\right)}{e^{z}-a_5}\right),\text{In}\left(e^{z}-a_5\right)\Big) \tag{2.14}$$

$$e^{a_6A_6}u\left(e^{x},e^{y},e^{y}\right)=u\left(x-\text{In}\left(1-a_6e^{z}\right),y,z-\text{In}\left(1-a_6e^{z}\right)\right) \tag{2.15}$$

Therefore we easily get

$$e^{a_6A_6}e^{a_5A_5}e^{a_4A_4}e^{a_3A_3}e^{a_2A_2}e^{a_1A_1}u\left(e^{x},e^{y},e^{y}\right)=$$

$$\upsilon\left\{z-\text{In}\left\{\left(e^{z}(1+a_5a_6)-a_5\right)\left(1-a_4e^{y}\right)-a_4a_5e^{x+y}\right\}\right\}u(\xi,n,p) \tag{2.16}$$

where

$$\xi=x+y+z-\text{In}\left(\frac{\left(1-a_6e^{z}\right)\left(1-a_4e^{y}\right)+a_4e^{x+y}}{\left(e^{z}(1+a_5a_6)-a_5\right)\left(e^{y}(1+a_2a_4)-a_4\right)+a_4e^{x+y}(1+a_2a_4)}\right)$$

$$\eta=a_1+\frac{\left\{e^{y}(1+a_3a_4)-a_3\right\}\left\{e^{z}(1-a_5a_6)-a_5\right\}+a_5e^{x+y}(1+a_3a_4)}{\left\{e^{z}(1-a_5a_6)-a_5\right\}\left(1-a_4e^{y}\right)-a_4a_5e^{x+y}}$$

$$\rho = a_2 + \frac{\{e^z(1+a_5a_6)-a_5\}(1-a_4e^y)-a_4a_5e^{x+y}}{(1-a_6e^z)(1-a_4e^y)+a_4e^{x+y}}.$$

3. Generating Functions. From (2.2) $u(e^x, e^y, e^z) = {}_2F_1(-n,\alpha;\alpha;v^x)e^{ny}e^{\alpha z}$ is a solution of the systems :

$$\begin{cases} Lu = 0 \\ (A_1 - n)u = 0 \end{cases}; \begin{cases} Lu = 0 \\ (A_2 - \alpha)u = 0 \end{cases}; \begin{cases} Lu = 0 \\ (A_1 + A_2 - n - \alpha)u = 0 \end{cases}$$

and from (2.9), we easily get

$$(SL)_2F_1(-n,\alpha;v;e^x)e^{ny}e^{\alpha z} = (LS)_2F_1(-n,\alpha;v;e^x)e^{ny}e^{\alpha z} = 0,$$

where $S = e^{a_6A_6}e^{a_5A_5}e^{a_4A_4}e^{a_3A_3}e^{a_2A_2}e^{a_1A_1}$.

Therefore the transformation $S\left[{}_2F_1(-n,\alpha;\alpha;v^x)e^{ny}e^{\alpha z}\right]$ is also annulled by L.

By putting $a_1 = a_2 = 0$ in (2.16) and then using it, we get

$$e^{a_6A_6}e^{a_5A_5}e^{a_4A_4}e^{a_3A_3}\left({}_2F_1(-n,\alpha;v;e^x)e^{ny+\alpha z}\right) =$$

$$(\alpha - v - n)\operatorname{In}\{(1-a_5e^{-z}+a_5a_6)(1-a_4e^y)-a_4a_5e^{x+y-z}\} - \alpha\operatorname{In}\{(1-a_4e^z)(1-a_4e^y)+a_4e^{x+y}\}$$

$$+ n\operatorname{In}\left[\{(1+a_3a_4)-a_3e^{-y}\}(1-a_5e^{-z}+a_5a_6)+a_5e^{x-y}(1+a_3a_4)\right]$$

$$+ \operatorname{In}\ {}_2F_1\left[\begin{matrix}-n,\alpha; \\ v;\end{matrix}\ \frac{e^x}{\{(1-a_6e^z)(1-a_4e^y)+a_4e^{x+y}\}\{(1-a_5e^{-z}+a_5a_6)\{(1+a_3a_4)-a_2e^{-y}\} + a_5e^{x-z}(1+a_2a_4)\}}\right] + \alpha(y+z) \qquad (3.1)$$

Again

$$e^{a_6A_6}e^{a_5A_5}e^{a_4A_4}e^{a_3A_3}\left[{}_2F_1(-n,\alpha;\alpha;v^x)e^{ny+\alpha z}\right]$$

$$= \operatorname{In}\sum_{p=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{(a_6)^p}{p!}(\alpha-m)_p\frac{(a_5)^m}{m!}(v-\alpha)_m\frac{(a_4)^l}{l!}(v+n-k)_l\frac{(a_2)^k}{k!}(-n)_k$$

$$+ \operatorname{In}\ {}_2F_1\left[\begin{matrix}-(n-k+l),\alpha-m+p; \\ v;\end{matrix}\ e^x\right] + (n-k+l) + (\alpha-m+p) \qquad (3.2)$$

Equating (3.1) and (3.2), so we have

$$(\alpha - \upsilon - n)\text{In}\left\{\left(1 - a_5 e^{-z} + a_5 a_6\right)\left(1 - a_4 e^{y}\right) - a_4 a_5 e^{x+y-z}\right\} + \text{In}\left\{\left(1 - a_2 e^{-z}\right)\left(1 - a_4 e^{y}\right) + a_4 e^{x+y}\right\}$$

$$+ n\text{In}\left[\left\{\left(1 + a_2 a_4\right) - a_2 e^{-y}\right\}\left(1 - a_5 e^{-z} + a_5 a_6\right) + a_5 e^{x-z}\left(1 + a_2 a_4\right)\right]$$

$$+ \text{In}\ {}_2F_1\left[\begin{matrix} -n, \alpha; \\ \upsilon; \end{matrix} \frac{e^x}{\left\{\left(1 - a_6 e^{x}\right)\left(1 - a_4 e^{y}\right) + a_4 e^{x+y}\right\}\left[\left(1 - a_5 e^{-z} + a_5 a_6\right)\left\{\left(1 + a_3 a_4\right) - a_2 e^{-y}\right\} + a_5 e^{x-z}\left(1 + a_2 a_4\right)\right]}\right]$$

$$= \sum_{p=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty} \frac{(a_6)^p}{p!}(\alpha - m)_p \frac{(a_5)^m}{m!}(\upsilon - \alpha)_m \frac{(a_4)^l}{l!}(\upsilon + n - k)_l \frac{(a_2)^k}{k!}(-n)_k$$

$$+ \text{In}\ {}_2F_1\left[\begin{matrix} -(n-k+l), \alpha - m + p; \\ \upsilon; \end{matrix} e^x\right] + y(1-k) + -z(m-p) \tag{3.3}$$

The above generating function does not seen to appear before where from a large number of different generating relations (near and known) may be easily obtained by attributing different values to a_i's of which the generating relations.

Derivation of some generating functions involving Jacobi polynomials from the relation (2.19)

Now putting .

$$\alpha = 1 + \alpha + \beta + n, \qquad \upsilon = 1 + \alpha, \qquad e^x = \frac{1 - e^y}{2} \text{ in (3.3) and then}$$

Case 1. Letting $a_2 = a_4 = a_6 = 0,\ a_5 = 1$, and $e^{-z} = -2t$ we get,

$$(1 + 2t)^{\beta}\left\{1 + t\left(1 - e^{y}\right)\right\}^{n} P_n^{\alpha,\beta}\left[\frac{e^y + \left(1 + e^y\right)t}{1 + \left(1 + e^y\right)t}\right]$$

$$= \sum_{m=0}^{\infty} \frac{2^m}{m!}(-\beta - n)_m P_n^{(\alpha\alpha,-m)}\left(e^x\right)t^m \tag{3.4}$$

Case 2 : Putting $a_2 = a_4 = a_5 = 0, a_6 = 1$, and $e^z = t$, we obtain

$$(1-t)^{-\alpha-\beta-n-1} P_n^{(\alpha\alpha,\beta)}\left(\frac{e^y - t}{1-t}\right) = \sum_{p=0}^{\infty} \frac{1}{p!}(1 + \alpha + \beta + n)_p P_n^{(\alpha\alpha + p)} e^y t^p \tag{3.5}$$

REFERENCES

[1] A.K. Chongdar, Group study for certain generating functions. *Cal. Math. Soc.*, **77** (1985), 151-157.

[2] E.B. Mc Bride, *Obtainging Generating Functions*. Springer Verlag, Berlin (1971).

[3] L. Weisner, Group theoretical origins of certain generating functions, *Pacific J. Math.*, 5 (1955), 1033-1039.

Jñānābha, Vol. 38, 2008

UNSTEDAY INCOMPRESSIBLE *MHD* HEAT TRANSER FLOW THROUGH A VARIABLE POROUS MEDIUM IN A HORIZONTAL POROUS CHANNEL UNDER SLIP BOUNDARY CONDITIONS

By

N.C.Jain

Department of Mathematics, University of Rajasthan

Jaipur-302004 ,Rajasthan, india

and

Dinesh Kumar Vijay

Department of Mathematics, Kautilya Institute of Technology & Engineering

Jaipur-302022 ,Rajasthan, india

E-mail:jainnc181@rediffmail.com

(Received : December 30, 2007*)*

ABSTRACT

The paper examines the problem of unsteady *MHD* flow and heat transfer of a viscous incompressible fluid in slip flow regime with variable permeability bounded by two parallel porous plates. Using perturbation technique the expressions are obtained for velocity and temperature distributions, skin friction and Nusselt number. The Effects of slip parameters, Hartmann number, Reynolds number, permeability parameter are discussed on velocity and temperature distributions. Important parameters, the skin friction and Nusselt number at both the plates are also calculated.

2000 Mathematics Subject Classification : Primary 76D, Secondary 76S05

Keywords : Unsteady, Variable permeability, Slip parameters, *MHD*, Porous Medium, Heat source/sink.

1. Introduction. Fluid flow and hєat transfer in porous media is an important subject in hydrology. It is of vital interest in petrolium and chemical engineerings. To study the underground water resources and seepage of water in a dam, one needs to investigate the flows through a porous media. Flow in a channel is very fundamental problem and attracted the attention of many research workers. Schlichting and Gersten [10], Bansal [4] and some others have considered the problem in their books. Considering magnetic effect Attia and Kolb [1], Yen and Chang [15] and Singh [11] studied the flow between two parallel plates. Jain and Bansal [5] considered temperature dependent viscosity in the Couette flow. Attia [2] studied hall current effects on the velocity and temperature fields of an unsteady Hartmann flow. Numerical solution of free convection *MHD* micropolar fluid flow between two parallel porous vertical plates is also discussed by Bhargava et al. [3].

Pulsatile flow in a channel or in a tube has application in Biofluid flow in the dialysis of blood in artificial kidneys. Sharma and Mishra [12] , Sarangi and Sharma [14] and Sharma et al. [13] have solved the problems with time dependent pressure gradiant.

In geothermal region situation may arise when the flow becomes unsteady and slip at the boundary may take place as well. At high altitude, the study of slip flow becomes very important. Kepping this in mind Soundlegekar and Arnake [9]. Johri and Sharma [6], Jain [7] and Jothimani and Anjali Devi [8] considered the slip flow boundary conditions in their problems.

The aim of present paper is to investigate the *MHD* flow and heat transfer of a viscous incompressible fluid in slip flow regime with variable permeability bounded by two parallel porous plates. It is found that the skin friction at both the plates increases with increase of slip parameter h_1.

2. Formulation of the Problem. Let us consider the unsteady incompressible *MHD* heat teransfer flow through a porous medium of variable permeability $K(t) = K_0\left(1+\varepsilon e^{-nt}\right)$ in slip flow regime bounded by two infinite long parallel thin porous plates. The parallel plates are placed at a distance h apart. Let x^*-axis be taken along the direction of plates and y^*-axis is taken in normal direction to the plates.

The equations of continuity, motion and energy for unsteady flow of an incompressible viscous fluids are

$$\frac{\partial \upsilon}{\partial t} = 0 \Rightarrow \upsilon = -\upsilon_0\left(1 + A\varepsilon e^{-nt}\right) \quad ...(1)$$

$$\rho\left(\frac{\partial \upsilon}{\partial t} + \frac{\partial u}{\partial y}\right) = \frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2} + \sigma B_0^2 u - \frac{u\mu}{K_0\left(1+\varepsilon e^{-nt}\right)} \quad ...(2)$$

$$\rho\frac{\partial \upsilon}{\partial t} = -\frac{\partial p}{\partial y} \quad ...(3)$$

$$\rho C_p\left(\frac{\partial T}{\partial t} + \frac{\partial T}{\partial y}\right) = k\frac{\partial^2 T}{\partial y^2} + \mu\left(\frac{\partial u}{\partial y}\right)^2 + Q(T - T_s) \quad ...(4)$$

The boundary conditions are

$$\left.\begin{aligned} y = 0: &\quad u = 0, &\quad T = T_0 \\ y = h: &\quad u = u_0\left(1 + A\varepsilon e^{nt}\right) + L_1\frac{du}{dy}, &\quad T = T_0 + L_2\frac{dT}{dy} \end{aligned}\right\} \quad ...(5)$$

where u, υ are the components of velocity along x-axis and y-axis respectively, t the time, υ_0 is the cross flow velocity, p the pressure, ρ is the density, C_p the specfic heat at constant pressure, k the thermal conductivity, μ the coefficient of viscocity,

Q the volumetric rate of heat generation, T the fluid temperature, T_s the static temperature, σ coefficient of the electrical conductivity, B_0 the coefficient of electromagnetic induction, here $L_1 = \left(\frac{2-m_1}{m_1}\right)L, L$ being mean free path and m_1 the Maxwell' s reflection coefficient.

A and n are real positive coefficients such that $\varepsilon A < 1$

Now introducing the following non-dimensional quantities

$$x^* = \frac{x}{h}, \quad y^* = \frac{y}{h}, \quad u^* = \frac{uh}{\upsilon}, \quad t^* = \frac{t\upsilon}{h^2}, \quad \Pr = \frac{\mu C_p}{k}, \quad p^* = \frac{ph^2}{\rho\upsilon^2},$$

$$\mathrm{Re} = \frac{V_o h}{\upsilon}, \quad K^* = \frac{K_0}{h^2}, \quad \theta = \frac{T-T_s}{T_0-T_s}, \quad E_c = \frac{\upsilon^2}{h^2 C_p (T_0 - T_s)}, \quad \alpha = \frac{Qh^2}{\upsilon C_p},$$

$$M^2 = \frac{\sigma B_0^2 h^2}{\rho\upsilon}, \quad n^* = \frac{nh^2}{\upsilon}.$$

Equations (3) and (4) reduce to the following form after dropping the asterisks over them

$$\frac{\partial u}{\partial t} - \mathrm{Re}\left(1 + A\varepsilon e^{-nt}\right)\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} - M^2 u - \left[\frac{1}{K\left(1 + A\varepsilon e^{-nt}\right)}\right]u \qquad \text{...(6)}$$

$$\Pr - \frac{\partial\theta}{\partial t} - \Pr\mathrm{Re}\left(1 + A\varepsilon e^{-nt}\right)\frac{\partial\theta}{\partial y} = -\frac{\partial^2\theta}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2 \Pr E_c + \Pr\alpha\theta \qquad \text{...(7)}$$

with corresponding boundary conditions

$$\left.\begin{array}{lll} u = 0 & \theta = 1 & at\ y = 0 \\ u = R_0\left(1 + A\varepsilon e^{-nt}\right) + h_1 \frac{du}{dy}, & \theta = 1 + h_2 \frac{d\theta}{dy} & at\ y = 1 \end{array}\right\} \qquad \text{...(8)}$$

Here $h_1 = \frac{L_1}{h}, h_2 = \frac{L_2}{h}, R_0 = \frac{hu_0}{\upsilon}$.

3. Solution of the Problem. We assume

$$\left.\begin{array}{l} \frac{\partial p}{\partial x} = \left(1 + A\varepsilon e^{-nt}\right) \\ u = u_0(y) + A\varepsilon e^{-nt} u_1(y) \\ \theta = \theta_0(y) + \varepsilon e^{-nt}\theta_1(y) \end{array}\right\} \qquad \text{...(9)}$$

Using equation (9) in the equations (6) and (7) and equating the coefficients of the some powers of ϵ, we get the following set of ordinary differential equations after neglecting the coefficients of $o(\varepsilon^2)$.

$$u_0'' + \operatorname{Re} u_0' - \left(M^2 + 1/K\right) u_0 = 1 \qquad \text{...(10)}$$

$$u_1'' + \operatorname{Re} u_1' - \left(M^2 + 1/K - n\right) u_1 = 1 - \operatorname{Re} u_0' - 1/K\, u_0 \qquad \text{...(11)}$$

$$\theta_0'' + \Pr \operatorname{Re} \theta_0' + \Pr \alpha \theta_0 = -\Pr E_c \left(u_0'\right)^2 \qquad \text{...(12)}$$

$$\theta_1'' + \Pr \operatorname{Re} \theta_1' + \Pr n \theta_1 + \Pr \alpha \theta_1 = -\Pr \operatorname{Re} A \theta_0' - 2 \Pr E_c u_0' u_1' \qquad \text{...(13)}$$

with corresponding boundary conditions becomes

$$\left.\begin{array}{llll} y = 0: \; u_0 = 0, & u_1 = 0, & \theta_0 = 1, & \theta_1 = 0 \\ y = 1: \; u_0 = R_0 + h_1 u_0', & u_1 = R_0 + h_1 u_1', & \theta_0 = 1 + h_2 \theta_0', & \theta_1 = h_2 \theta_1' \end{array}\right\} \qquad \text{...(14)}$$

Equations (10) to (13) are second order linear differential equations with constant coefficients, the solution of which are

$$u = C_1 e^{m_1 y} + C_2 e^{m_2 y} - A_1 + A\varepsilon e^{-nt}\left(C_3 e^{m_3 y} + C_4 e^{m_4 y} + A_4 e^{m_1 y} + A_5 e^{m_2 y} + A_6\right) \qquad \text{...(15)}$$

$$\begin{aligned} \theta = {} & C_5 e^{m_5 y} + C_6 e^{m_6 y} + A_7 e^{2m_1 y} + A_8 e^{2m_2 y} + A_9 e^{(m_1 + m_2)y} + \varepsilon e^{-nt} \\ & \left(C_7 e^{m_7 y} + C_8 e^{m_8 y} + A_{10} e^{m_5 y} + A_{11} e^{m_6 y} + A_{12} e^{2m_1 y} + A_{13} e^{2m_2 y}\right. \\ & A_{14} e^{(m_1 + m_2)y} + A_{15} e^{(m_1 + m_3)y} + A_{16} e^{(m_1 + m_4)y} + A_{17} e^{2m_1 y} + A_{18} e^{(m_1 + m_2)y} \\ & \left.A_{19} e^{(m_2 + m_3)y} + A_{20} e^{(m_2 + m_4)y} + A_{21} e^{(m_1 + m_2)y} + A_{22} e^{2m_2 y}\right) \end{aligned} \qquad \text{...(16)}$$

where

$$A_1 = M^2 + 1/K \qquad m_1 = \frac{-\operatorname{Re} + \sqrt{\operatorname{Re}^2 + 4\left(M^2 + 1/K\right)}}{2},$$

$$m_2 = \frac{-\operatorname{Re} - \sqrt{\operatorname{Re}^2 + 4\left(M^2 + 1/K\right)}}{2}, \qquad C_1 = \frac{A_1 e^{m_2}\left(1 - h_1 m_2\right) - A_1 - R_0}{e^{m_2}\left(1 - h_1 m_2\right) - e^{m_1}\left(1 - h_1 m_1\right)},$$

$$C_2 = \frac{A_1\left[1 - e^{m_1}\left(1 - h_1 m_1\right)\right] + R_0}{e^{m_2}\left(1 - h_1 m_2\right) - e^{m_1}\left(1 - h_1 m_1\right)}, \quad m_3 = \frac{-\operatorname{Re} + \sqrt{\operatorname{Re}^2 - 4\left(n - A_1\right)}}{2},$$

$$m_4 = \frac{-\operatorname{Re} - \sqrt{\operatorname{Re}^2 - 4\left(n - A_1\right)}}{2}, \qquad A_2 = -\left(\operatorname{Re} C_1 m_1 + \frac{C_1}{K}\right),$$

$$A_3 = -\left(\operatorname{Re} C_2 m_2 + \frac{C_2}{K}\right), \qquad A_4 = \frac{A_2}{m_1^2 + \operatorname{Re} m_1 + \left(n - A_1\right)},$$

$$A_5 = \frac{A_3}{m_2^2 + \operatorname{Re} m_2 + \left(n - A_1\right)}, \qquad A_6 = \frac{1 + A_1}{n - A_1}, \qquad C_3 = \left(C_4 + A_4 + A_5 + A_6\right),$$

$$C_4 = \frac{R_0 - A_6 + A_4 e^{m_1}\left(h_1 m_1 - 1\right) + A_5 e^{m_2}\left(h_1 m_2 - 1\right) - e^{m_3}\left(1 - h_1 m_3\right)\left(A_4 + A_5 + A_6\right)}{e^{m_4}\left(1 - h_1 m_4\right) - e^{m_3}\left(1 - h_1 m_3\right)},$$

$$m_5 = \frac{-PrRe + \sqrt{Pr^2Re^2 - 4Pr\alpha}}{2}, \qquad m_6 = \frac{-PrRe - \sqrt{Pr^2Re^2 - 4Pr\alpha}}{2},$$

$$A_7 = -\frac{PrE_c m_1^2 C_1^2}{4m_1^2 + 2Pr\,\mathrm{Re}\, m_1 + Pr\alpha}, \qquad A_8 = -\frac{PrE_c m_2^2 C_2^2}{4m_2^2 + 2Pr\,\mathrm{Re}\, m_2 + Pr\alpha},$$

$$A_9 = -\frac{2m_1 m_2 C_1 C_2}{(m_1+m_2)^2 + \mathrm{Pr\,Re}(m_1+m_2) + \mathrm{Pr}\,\alpha}, \quad C_5 = 1 - (C_6 + A_7 + A_8 + A_9),$$

$$C_6 = \frac{1 + h_2\left[2m_1 A_7 e^{2m_1} + 2m_2 A_8 e^{2m_2} + (m_1+m_2) A_9 e^{(m_1+m_2)}\right] - e^{m_5}(1-h_2 m_5)(1 - A_7 - A_8 - A_9)}{e^{m_6}(1-h_2 m_6) - e^{m_5}(1-h_2 m_5)},$$

$$m_7 = \frac{-PrRe + \sqrt{Pr^2Re^2 - 4Pr(n+\alpha)}}{2}, \qquad m_8 = \frac{-PrRe - \sqrt{Pr^2Re^2 - 4Pr(n+\alpha)}}{2},$$

$$A_{10} = -\frac{PrReAm_5C_5}{m_5^2 + PrRem_5 + Pr(n+\alpha)}, \qquad A_{11} = -\frac{PrReAm_6C_6}{m_6^2 + PrRem_6 + Pr(n+\alpha)},$$

$$A_{12} = -\frac{2PrReAm_1A_7}{4m_1^2 + 2PrRem_1 + Pr(n+\alpha)}, \qquad A_{13} = -\frac{2PrReAm_2A_8}{4m_2^2 + 2PrRem_2 + Pr(n+\alpha)},$$

$$A_{14} = -\frac{PrReAA_9(m_1+m_2)}{(m_1+m_2)^2 + PrRe(m_1+m_2) + Pr(n+\alpha)},$$

$$A_{15} = -\frac{2PrE_cC_1C_3m_1m_3}{(m_1+m_3)^2 + PrRe(m_1+m_3) + Pr(n+\alpha)},$$

$$A_{16} = -\frac{2PrE_cC_1C_4m_1m_4}{(m_1+m_4)^2 + PrRe(m_1+m_4) + Pr(n+\alpha)},$$

$$A_{17} = -\frac{2PrE_cC_1m_1^2A_4}{4m_1^2 + 2PrRem_1 + Pr(n+\alpha)}, \quad A_{18} = -\frac{2PrE_cC_1A_5m_1m_2}{(m_1+m_2)^2 + PrRe(m_1+m_2) + Pr(n+\alpha)},$$

$$A_{19} = -\frac{2PrE_cC_2C_3m_2m_3}{(m_2+m_3)^2 + PrRe(m_2+m_3) + Pr(n+\alpha)},$$

$$A_{20} = -\frac{2PrE_cC_2C_4m_2m_4}{(m_2+m_4)^2 + PrRe(m_2+m_4) + Pr(n+\alpha)},$$

$$A_{21} = -\frac{2PrE_cC_2A_4m_1m_2}{(m_1+m_2)^2 + PrRe(m_1+m_2) + Pr(n+\alpha)}, \quad A_{22} = -\frac{2PrE_cC_2A_5m_2^2}{4m_2^2 + 2PrRem_2 + Pr(n+\alpha)},$$

$$C_7 = -\left[C_8 + A_{10} + A_{11} + A_{12} + A_{13} + A_{14} + A_{15} + A_{16} + A_{17} + A_{18} + A_{19} + A_{20} + A_{21} + A_{22}\right],$$

$$h_2\left[m_5A_{10}e^{m_5} + m_6A_{11}e^{m_6} + 2m_1A_{12}e^{2m_1} + 2m_2A_{13}e^{2m_2} + (m_1+m_2)A_{14}e^{(m_1+m_2)}\right.$$

$$+(m_1+m_3)A_{15}e^{(m_1+m_3)}+(m_1+m_4)A_{16}e^{(m_1+m_4)}+2m_1A_{17}e^{2m_1}+(m_1+m_2)A_{18}e^{(m_1+m_2)}$$

$$+(m_1+m_3)A_{19}e^{(m_1+m_3)}+(m_2+m_4)A_{20}e^{(m_2+m_4)}+(m_1+m_2)A_{21}e^{(m_1+m_2)}+2m_2A_{22}e^{2m_2}\Big]$$

$$-\Big[A_{10}e^{m_5}+A_{11}e^{m_6}+A_{12}e^{2m_1}+A_{13}e^{2m_2}+A_{14}e^{(m_1+m_2)}+A_{15}e^{(m_1+m_3)}+A_{16}e^{(m_1+m_4)}$$

$$+A_{17}e^{2m_1}+A_{18}e^{(m_1+m_2)}+A_{19}e^{(m_2+m_3)}+A_{20}e^{(m_2+m_4)}+A_{21}e^{(m_1+m_2)}+A_{22}e^{2m_2}\Big]$$

$$C_8=\frac{\begin{array}{l}+e^{m_7}(1-h_2m_7)\big[A_{10}+A_{11}+A_{12}+A_{13}+A_{14}+A_{15}+A_{16}+A_{17}+A_{18}\\ +A_{19}+A_{20}+A_{21}+A_{22}\big]\end{array}}{e^{m_8}(1-h_2m_8)-e^{m_7}(1-h_2m_7)},$$

4. Skin-Friction. The coefficient of skin friction at the lower plate is

$$(C_f)_{y=0}=\left(\frac{\partial u}{\partial y}\right)_{at\ y=0}$$

$$=u'_0+A\varepsilon e^{-nt}u'_1$$

$$=m_1C_1+m_2C_2+A\varepsilon e^{-nt}\left[m_3C_3+m_4C_4+m_1A_4+m_2A_5\right]$$

The coefficient of skin friction at the lower plate is

$$(C_f)_{y=1}=\left(\frac{\partial u}{\partial y}\right)_{at\ y=1}$$

$$=m_1C_1e^{m_1}+m_2C_2e^{m_2}+A\varepsilon e^{-nt}\left[m_3C_3e^{m_3}+m_4C_4e^{m_4}+m_1A_4e^{m_1}+m_2A_5e^{m_2}\right]$$

5. Nusselt Number. The rate of heat transfer in terms of Nusselt number at the lower plate is

$$(Nu)_{y=0}=\left(\frac{\partial\theta}{\partial y}\right)_{y=0}$$

$$=m_5C_5+m_6C_6+2m_1A_7+2m_2A_8+(m_1+m_2)A_9+\varepsilon e^{-nt}\big[m_7C_7+m_8C_8+m_5A_{10}$$

$$+m_6A_{11}+2m_1A_{12}+2m_2A_{13}+(m_1+m_2)A_{12}+(m_1+m_3)A_{15}+(m_1+m_4)A_{16}$$

$$+2m_1A_{17}+(m_1+m_2)A_{18}+(m_2+m_3)A_{19}+(m_2+m_4)A_{20}+(m_1+m_2)A_{21}+2m_2A_{22}\big]$$

The rate of heat transfer in terms of Nusselt number at the upper plate is

$$(Nu)_{y=1}=\left(\frac{\partial\theta}{\partial y}\right)_{y=1}$$

$$=m_2C_5e^{m_5}+m_6C_6e^{m_6}+2m_1A_7e^{2m_1}+2m_2A_8e^{2m_2}+(m_1+m_2)A_9e^{(m_1+m_2)}+\varepsilon e^{-nt}$$

$$\Big[m_7C_7e^{m_7}+m_8C_8e^{m_8}+m_5A_{10}e^{m_5}+m_6A_{11}e^{m_6}+2m_1A_{12}e^{2m_1}+2m_2A_{13}e^{2m_2}$$
$$+(m_1+m_2)A_{14}e^{(m_1+m_2)}+(m_1+m_3)A_{15}e^{(m_1+m_3)}+(m_1+m_4)A_{16}e^{(m_1+m_4)}$$
$$+2m_1A_{17}e^{2m_1}+(m_1+m_2)A_{18}e^{(m_1+m_2)}+(m_2+m_3)A_{19}e^{(m_2+m_3)}$$
$$+(m_2+m_4)A_{20}e^{(m_2+m_4)}+(m_1+m_2)A_{21}e^{(m_1+m_2)}+2m_2A_{22}e^{2m_2}\Big].$$

6. Result and Discussion. In order to understand the solutions physically, we have calculated the numerical values of the velocity distributions [Figure 1.0], temperature distributions [Figure 2.0], skin friction [Figure 3.0 and Figure 4.0] and Nusselt number [Figure 5.0 and Figure 6.0] for different values of h_1 (slip parameter), h_2 (temperature jump), M (Magnetic parameter), Re (Reynolds number), α (volumetric rate of heat generation) and K (Permeability parameter).

In Figure 1.0, the velocity distribution (u) is plotted against y. It is being observed that velocity increases with the increase in h_1, Re and Ro but decreases with increase in M and K. It is also observed that increase in velocity with the increase in injection parameter takes place for both the cases slip flow or no slip flow.

In Figure 2.0, temperature distribution (θ) is plotted against y. It is being seen that temperature increases as M, K, Pr and α increase but phenomena reverses for the case of h_1, h_2 and Ro. It is interesting to note that for the case of $Pr=7.0$ (water as a fluid), the temperature first increases and after some channel width it decreases asymptotically, while for the case of $Pr=0.71$ (air as a fluid) the temperature decreases continuously.

Skin friction which is plotted in Figures 3.0 and 4.0 against K is having a worth noting observation. It is being observed that skin friction at lower plate $(C_f)_{y=0}$ increases with the increase in Re and h_1 but decreases with the increase in M and t while skin friction at the upper plate $(C_f)_{y=1}$ increases with increase in M and h_1 but decreases with the increase in Re and t. Moreover, increase in K has very small increasing effect on $(C_f)_{y=0}$ but increase in K decreases $(C_f)_{y=1}$ sharply for lower values of K than for higher values.

Nusselt number at the lower and upper plates are plotted against t in Figures 5.0 and 6.0 respectively. From the figures it is observed that Nusselt number at both the plates increases with the increase in α and K. Moreover, Nusselt number (Nu) at the lower plate remains positive for $Pr=7.0$ but for the $Pr=0.71$ it is negative. As expected Nusselt number decreases for sink tha source at both the plates.

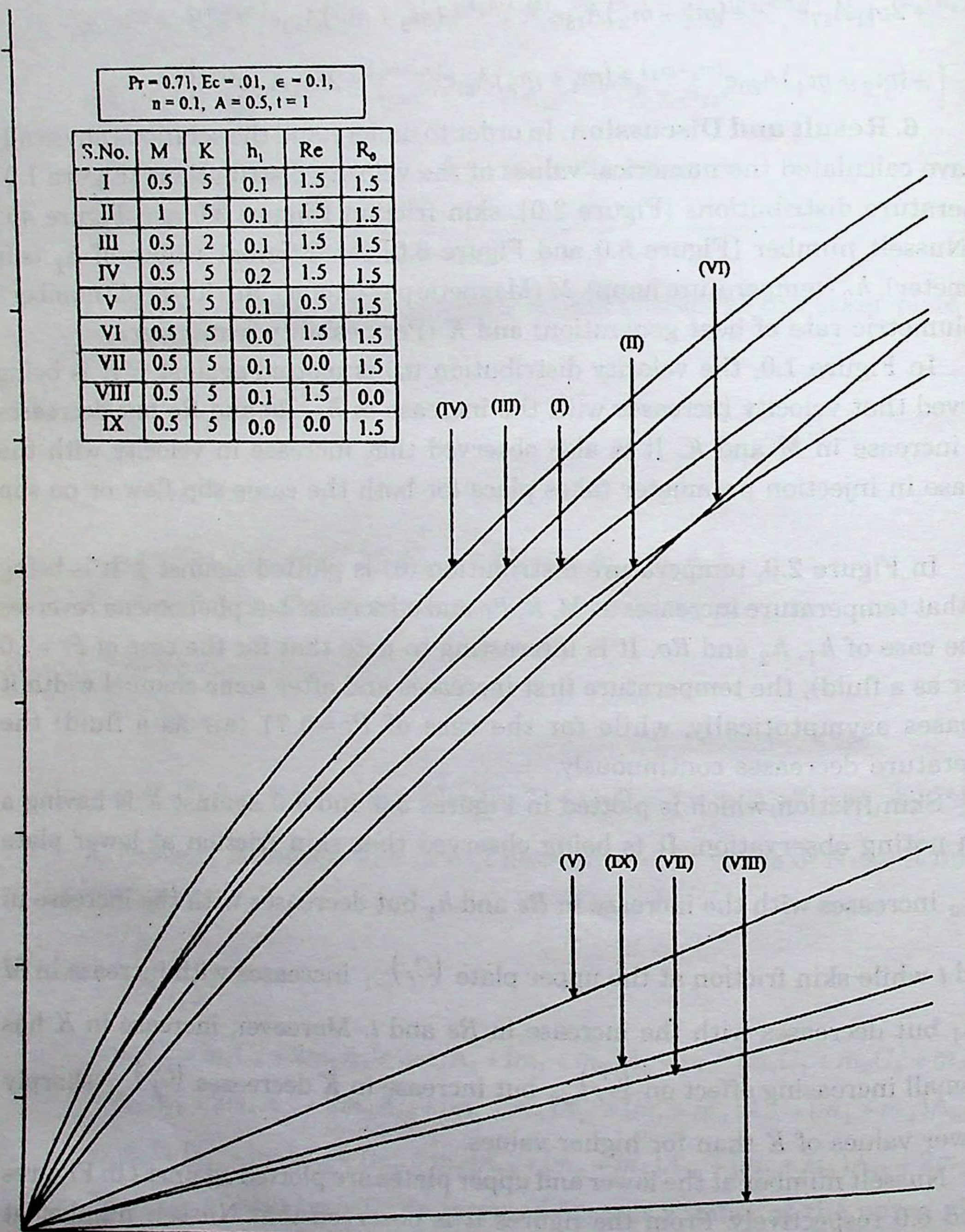
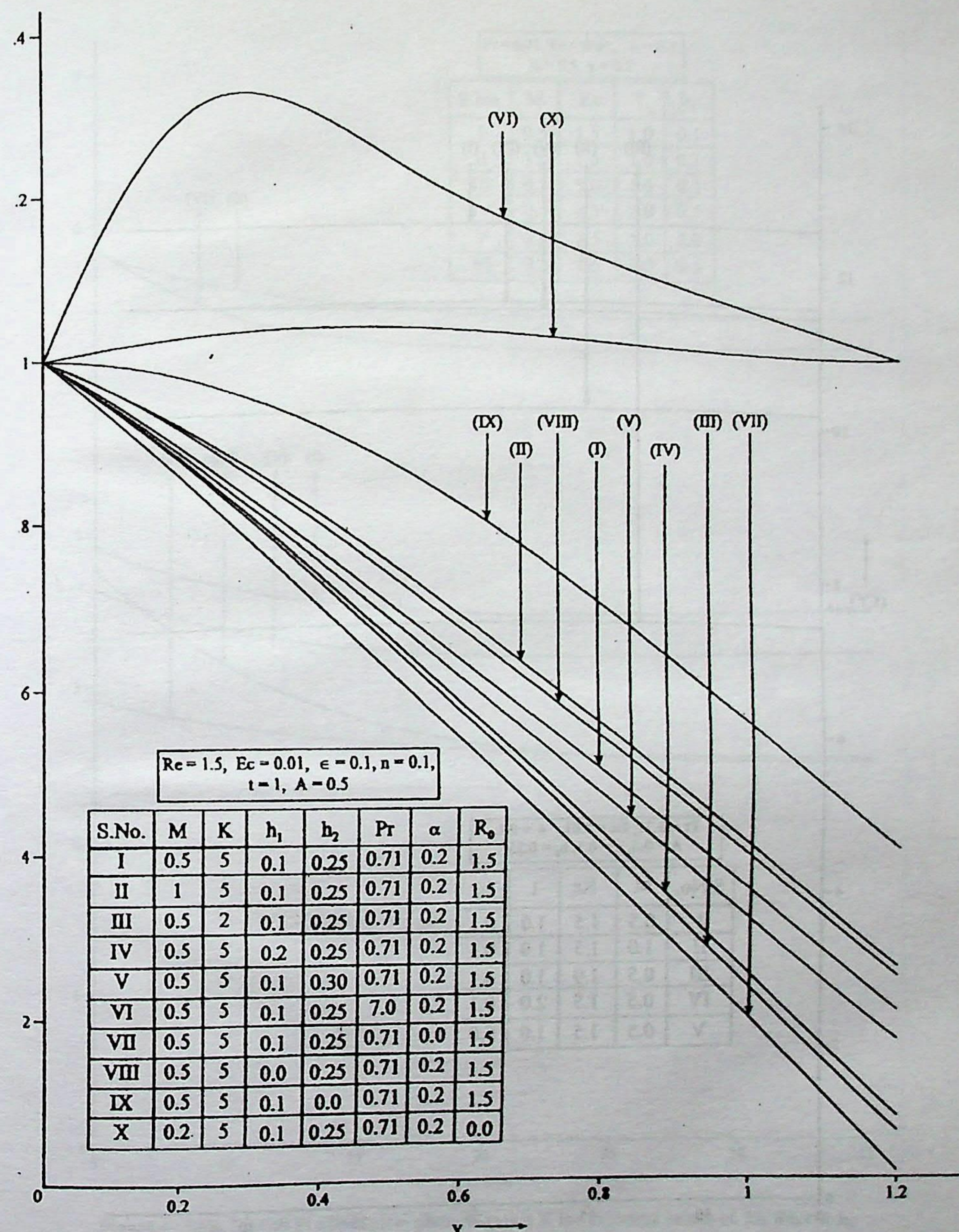
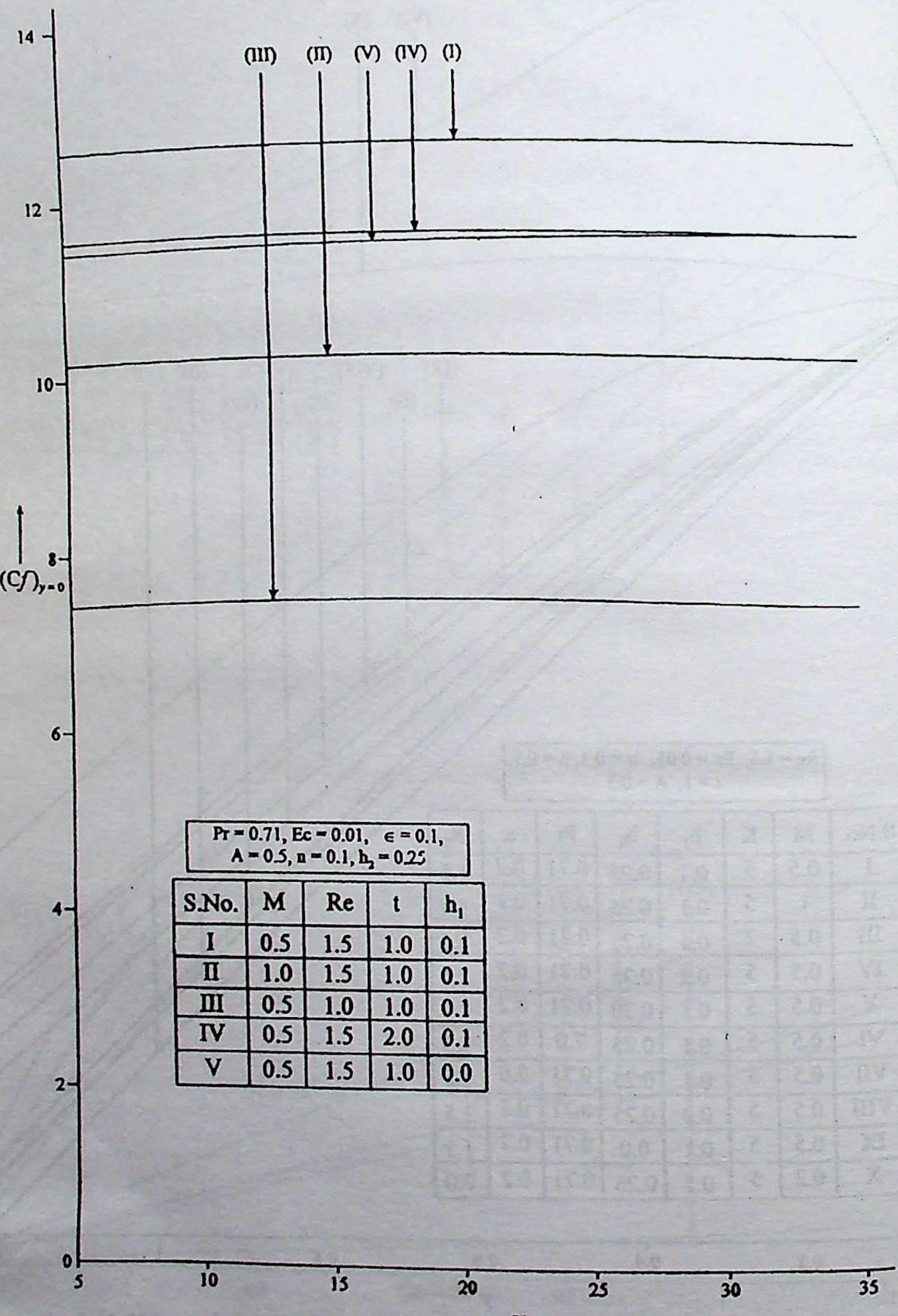


Figure 1 : Velocity distribution plotted against y for different values of M, K, h_1, Re, R_0.



S.No.	M	K	h_1	h_2	Pr	α	R_0
I	0.5	5	0.1	0.25	0.71	0.2	1.5
II	1	5	0.1	0.25	0.71	0.2	1.5
III	0.5	2	0.1	0.25	0.71	0.2	1.5
IV	0.5	5	0.2	0.25	0.71	0.2	1.5
V	0.5	5	0.1	0.30	0.71	0.2	1.5
VI	0.5	5	0.1	0.25	7.0	0.2	1.5
VII	0.5	5	0.1	0.25	0.71	0.0	1.5
VIII	0.5	5	0.0	0.25	0.71	0.2	1.5
IX	0.5	5	0.1	0.0	0.71	0.2	1.5
X	0.2	5	0.1	0.25	0.71	0.2	0.0

Figure 2 : Temperature distribution plotted against y for different values of M, K, h_1, h_2, α, R_0.



S.No.	M	Re	t	h_1
I	0.5	1.5	1.0	0.1
II	1.0	1.5	1.0	0.1
III	0.5	1.0	1.0	0.1
IV	0.5	1.5	2.0	0.1
V	0.5	1.5	1.0	0.0

Figure 3 : Skin fraction at lower plate plotted against K for different values of M, Re, t & h_1.

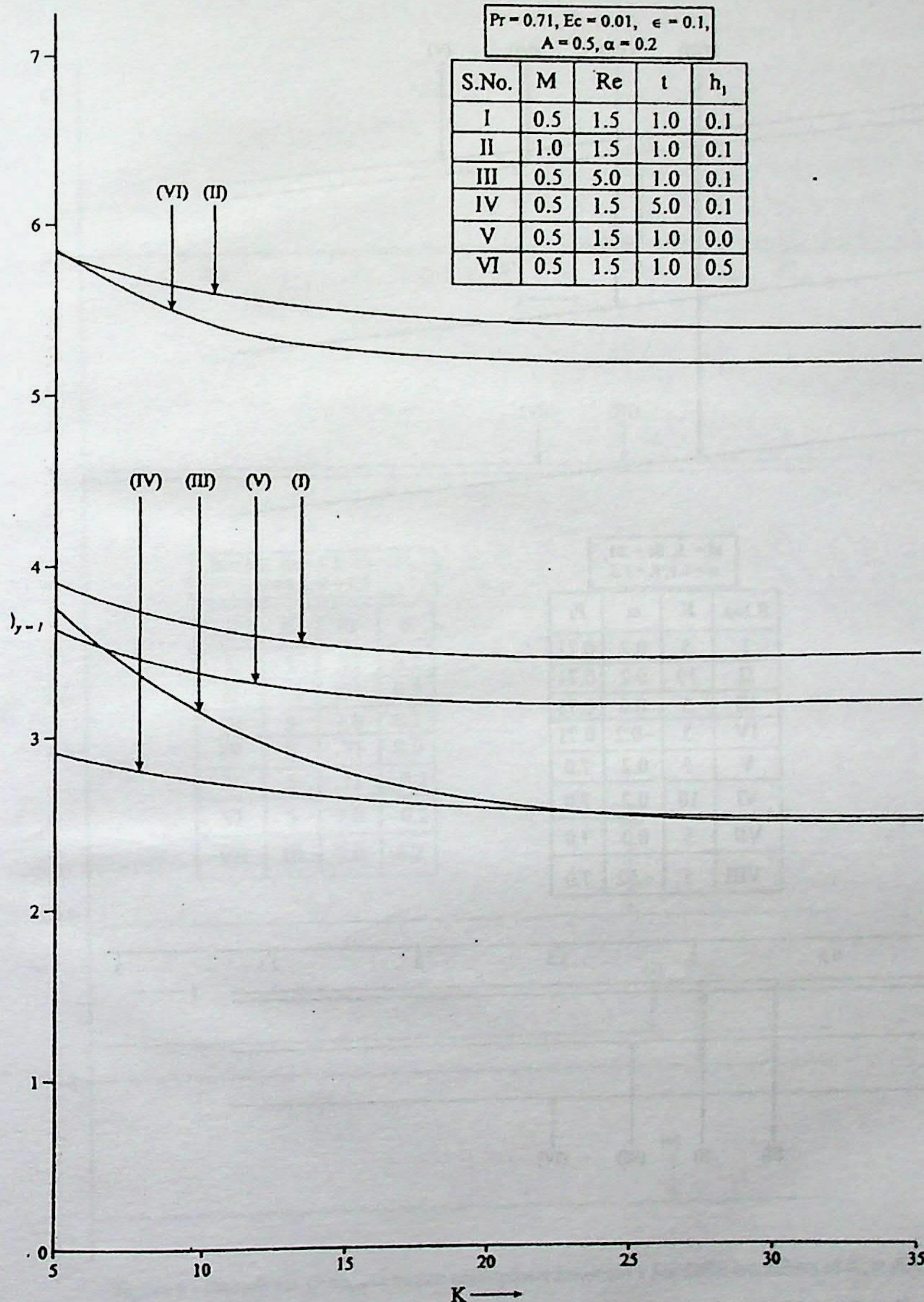


Figure 4 : Skin friction at upper plate plotted against K for different values of M, Re, t & h_1.

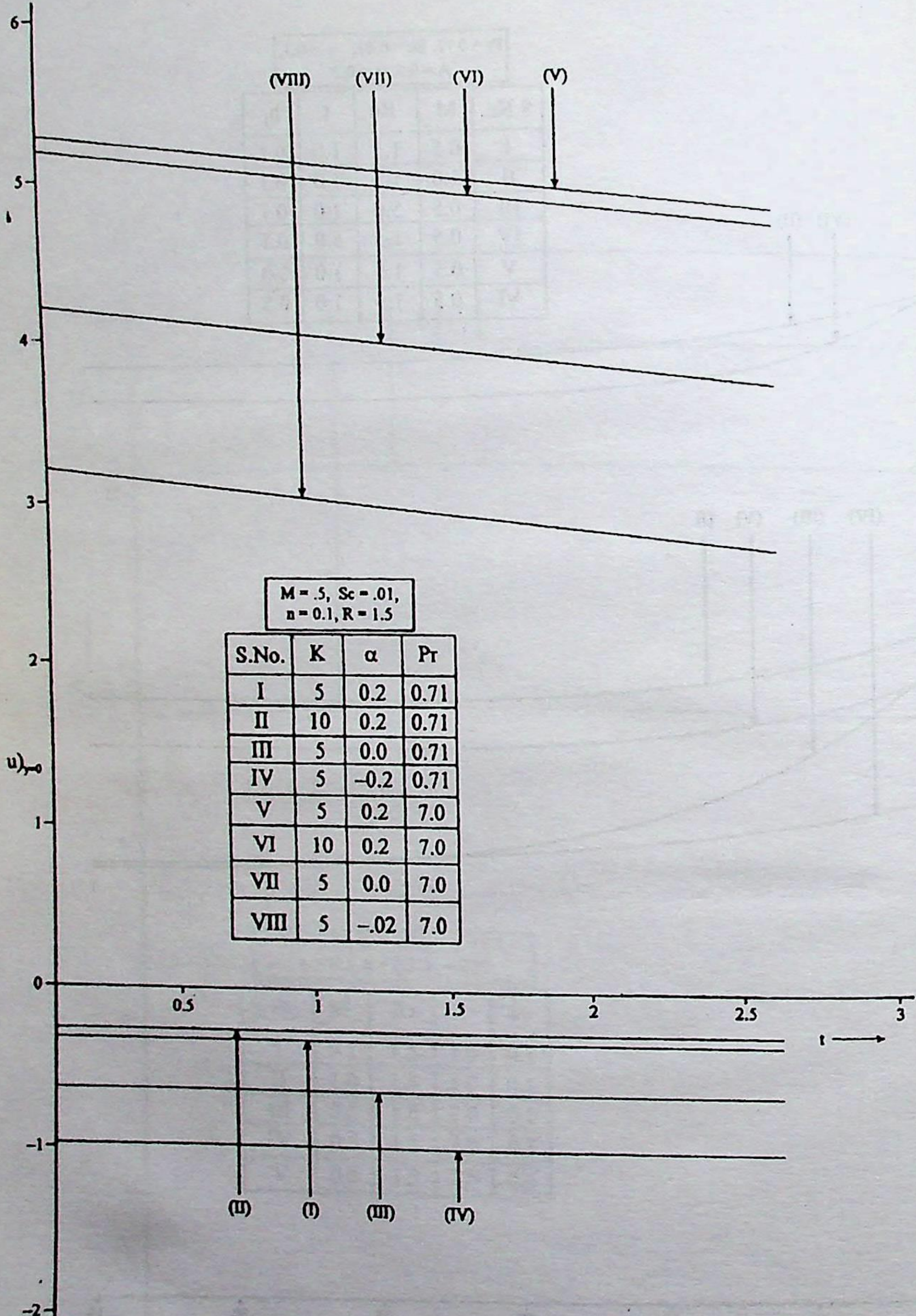


Figure 5 : Nusselt no. $(Nu)_{y=0}$ at lower plate plotted against t for different values of K, α & Pr.

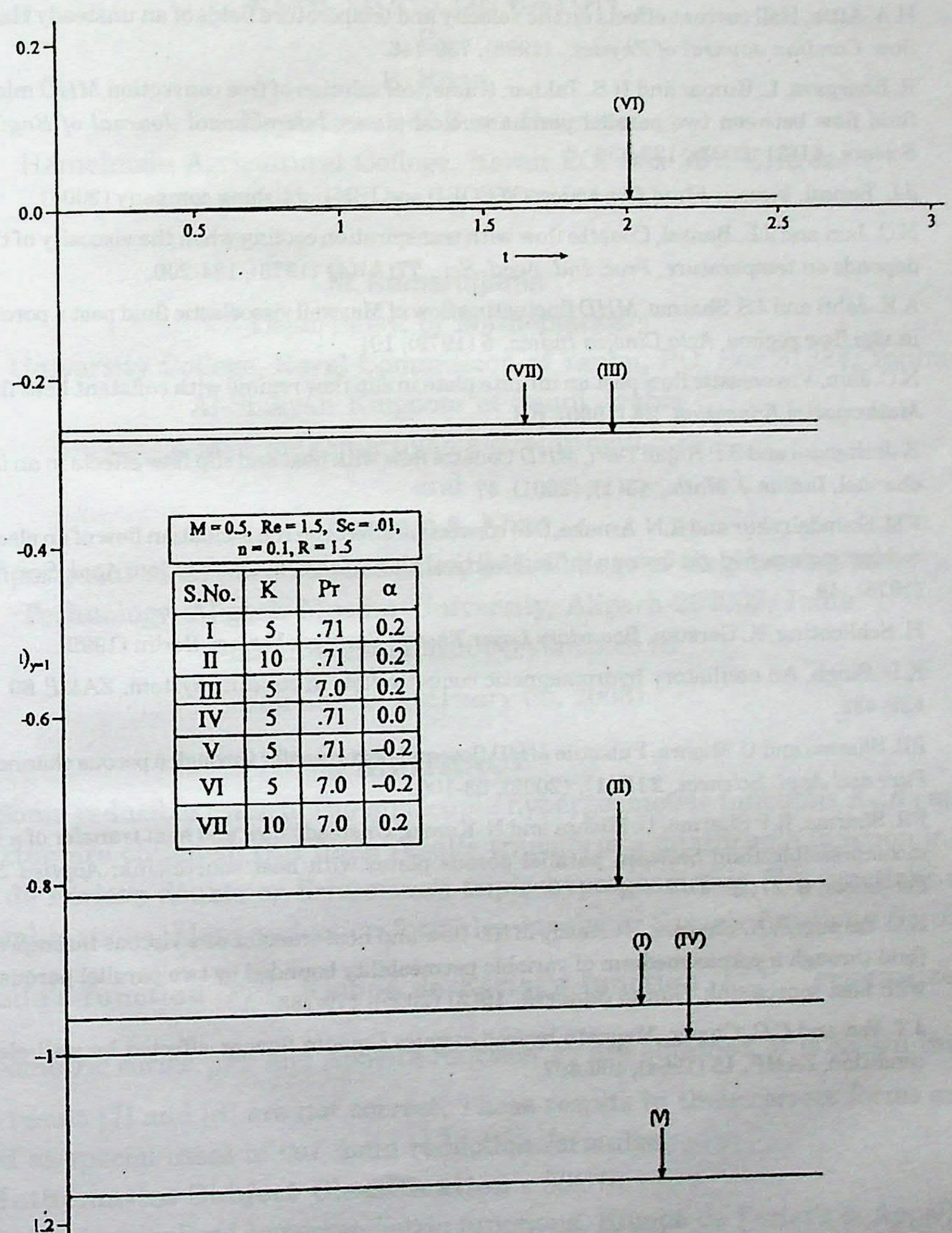


Figure 6 : Nusselt no. $(Nu)_{y=1}$ at upper plate plotted against t for different values of K, α & Pr.

REFERENCES

[1] H.A. Attia and N.A. Kotb, *MHD* flow between two parallel plates with heat transfer, *Acta Mech*, **117** (1996) 215-220.

[2] H.A. Attia, Hall current effects on the velocity and temperature fields of an unsteady Hartmann flow *Candian Jounral of Physics.*, (1998), 739-746.

[3] R. Bhargava, L. Kumar and H.S. Takhar, Numerical solution of free convection *MHD* micropolar fluid flow between two parallel porous vertical plates, *International Journal of Engineering Science*, **41(2)** (2003), 123-136.

[4] J.L. Bansal, *Viscous Fluid Dynamics,* OXFORD and IBH publishing company (2004).

[5] N.C. Jain and J.L. Bansal, Couette flow with transpiration cooling when the viscosity of the fluid depends on temperature, *Proc. Ind. Acad. Sci.,* **77(A)(4)** (1973), 184-200.

[6] A.K. Johri and J.S. Sharma, *MHD* fluctuating flow of Maxwell viscoelastic fluid past a porous plate in slip flow regime, *Acta Cinecia Indica,* **5** (1979), 101.

[7] N.C. Jain, Viscoelastic flow past an infinite plate in slip flow regime with constant heat flux, *The Mathematics Education,* **24** (1990), p.3.

[8] S. Jothimani and S.P. Anjali Devi, *MHD* Couette flow with heat and slip flow effects in an inclined channel, *Indian J. Math.,* **43(1),** (2001), 47

[9] V.M. Soundalgekur and R.N. Arnake, free convection effects on the oscillation flow of an electrically conducting rarefied gas over an infinite vertical plate with constant suction, *Appl. Sce. Res.*, 34 (1978), 49.

[10] H. Schlichting, K. Gersten, *Boundary Layer Theory,* Springer Verlage Berlin (1999).

[11] K.D. Singh, An oscillatory hydromagnetic coquette flow in rotating system, *ZAMP,* **80** (2000), 429-432.

[12] P.R. Sharma and U. Mishra, Pulsatile *MHD* flow and heat transfer through a porous channel, *Bull. Pure and Appl. Sciences,* **21E(1),** (2002), 93-100.

[13] P.R. Sharma, R.P. Sharma, U. Mishra and N. Kumar, Unsteady flow and heat transfer of a viscous incompressible fluid between parallel porous plates with heat source/sink, *Applied Science Periodical,* **6** (2) (2004).

[14] K.C. Sarangi, V.K. Sharma, Unsteady *MHD* flow and heat transfer of a viscous incompressible, fluid through a porous medium of variable permeability bounded by two parallel porous plates with heat source/sink, *Ganita Sandesh,* **19(2)** (2005), 179-188.

[15] J.T. Yen and C.C. Chang, Magneto hydrodynamics Couette flow as affected by wall electrical condition, *ZAMP*, **15** (1964), 400-407.

J ñānābha, Vol. 38, 2008

REDUCIBILITY OF THE QUARDUPLE HYPERGEOMETRIC FUNCTIONS OF EXTON

By

B. Khan

Department of Basic Sciences

Hamelmalo Agricultural College, Keran P.O. Box 397, Eritrea.

e-mail : bk_warsi@rediffmail.com

M. Kamarujjama

Department of Mathematics

Yanbu University College, Royal Commission of Yanbu, P.O. Box 31387, Yanbu Al-Sinayah Kingdom of Saudi Arabia

e-mail:mdkamarujjama@rediffmail.com

and

Nassem A. Khan

Department of Applied Mathematics, Z.H. College of Engineering and Technology, Aligarh Muslim University, Aligarh-202002, India

e-mail:naseemamr009@yahoo.co.in

(*Received : February* 05, 2008)

ABSTRACT

Some reducible cases of the quadruple hypergeometric functions K_1,K_2 and K_3 of Exton are obtained. Our main results transform a quardruple function into Kampé de Fériet's double or Srivastava's triple hypergeometric $F^{(3)}$ functions or their combinations. Many reduction formulae involving Saran's functions E_E, E_F Lauricella's function $F_C^{(3)}$, Kampé de Fériet's function $F^{p:q:r}_{s:t;}$, generalized hypergeometric series ${}_pF_q$ and Appell's function F_4 are obtained. It is shown that Exton's result [7] and [8] are not correct. These results in their correct forms are obtained as special cases of our main reduction formulae.

2000 Mathematics Subject Classification : 33C70.

Keywords : Generalized hypergeometric functions, Kmapé de Fériet's & Appell's functions, Quadruple hypergeometric functions, and Lauricella's function.

1. Introduction. In the year 1972, Exton [6]] defined and examined a few properties of quadruple hypergeometric functions. In his notations K_1, K_2 and K_3 are given by [8;p.78]

$$K_1[a,a,a,a;b,b,b,c;d,e,f,d;x,y,z,t] = \sum_{m,n,p,q=0}^{\infty} \frac{(a)_{m+n+p+q}(b)_{m+n+p}(c)_q x^m y^n z^p t^q}{(d)_{m+q}(e)_n(f)_p m!n!p!q!} \quad (1.1)$$

$$K_2[a,a,a,a;b,b,b,c;d,e,f,g;x,y,z,t] = \sum_{m,n,p,q=0}^{\infty} \frac{(a)_{m+n+p+q}(b)_{m+n+p}(c)_q x^m y^n z^p t^q}{(d)_m(e)_n(f)_p(g)_q m!n!p!q!} \quad (1.2)$$

and

$$K_3[a,a,a,a;b,b,c,c;d,e,e,d;x,y,z,t] = \sum_{m,n,p,q=0}^{\infty} \frac{(a)_{m+n+p+q}(b)_{m+n}(c)_{p+q} x^m y^n z^p t^q}{(d)_{m+q}(e)_{n+p} m!n!p!q!} \quad (1.3)$$

respectively, where

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}.$$

The convergence of the series K_1, K_2 and K_3 are investigated by Exton, by mean of the general theory obtained in [8, p. 65(2.9)].

In this paper, various reducible cases of the functions K_1, K_2 and K_3 are obtained. Our results transform a function of four variables into a double or triple series or their combinations. The main insterst of the result obtained in Section 2 is that the series of Lauricella's $F_C^{(3)}$ is tranformed into a combinations of Kampé de Fériet's functions. In Section 3 a reduction of F_E in terms of generalized hypergeometric series ${}_4F_3$ is given. In Section 4 a reduction of K_3 into a combination of four $F^{(3)}$'s is obtained which further gives a transformation of Saran's F_F into a combination of two Kampé de Fériet's functions.

In a paper of Exton [7, p. 66(3.2)], the reduction formula for Saran's function F_E [11] is given in the form

$$F_E[a,a,a;b_1,b_2,b_2;c_1,c_2,c_3;x,y,y] = F_{0:3,3}^{1:3;3}\left[\begin{matrix} a : b_1, A, B; b_2, \frac{c_2+c_3}{2}, \frac{c_2+c_3-1}{2} \\ - : b_2, A, B; c_2, c_3, c_2+c_3-1; \end{matrix} ; x, 4y\right], \quad (1.4)$$

where as in a recent book of Exton [8, p.134 (4.7.8)], the reduction of F_E is given by

$$F_E[a,a,a;b_1,b_2,b_2;c_1,c_2,c_3;x,y,y] = F_{0:3;3}^{1:3;3}\left[\begin{matrix} a : b_1, A, B; b_2, \frac{c_2+c_3}{2}, \frac{c_2+c_3-1}{2} \\ -; b_2, A, B; c_1, c_3, c_2+c_3-1; \end{matrix} ; x, 4y\right], \quad (1.5)$$

where $F_{0:3;3}^{1:3;3}$ is Kampé de Fériet's function in contracted notation of Burchnall and Chaundy [2;p. 112-113]. In equation (1.4) and (1.5), A, and B are arbitrary

parameters in which numerator parameters are cancelled by denominator parameters A and B.

It is to be noted that (1.4) and (1.5) are incorrect. In fact, their correct form is obtained in Section 2 and follows as a special case of our reduction formula of K_1.

2. Reducibility of K_1-Function. From (1.1), we have

$$K_1[a,a,a,a;b,b,b,c;d,e,f,d;x,y,z,t]=\sum_{m,q=0}^{\infty}\frac{(a)_{m+q}(b)_m(c)_q x^m t^q}{(d)_{m+q}m!q!}$$

$$F_4[a+m+q;b+m;e,f;y,z], \tag{2.1}$$

where F_4 is Appell's function of fourth kind [1,p.14(14)].

Putting $z=y$ in (2.1) and using a result of Burchnall [3; p.101 (37)], we get

$$K_1[a,a,a,a;b,b,b,c;d,e,f,d;x,y,y,t]=\sum_{m,q=0}^{\infty}\frac{(a)_{m+q}(b)_m(c)_q x^m t^q}{(d)_{m+q}m!q!}$$

$${}_4F_3\left[\begin{matrix}a+m+q,b+m,\dfrac{e+f}{2},\dfrac{e+f-1}{2};4y\\ e,f,e+f-1\qquad\qquad\qquad ;\end{matrix}\right], \tag{2.2}$$

where ${}_4F_3$ is generalized hypergeometric function [10; p. 73(2)]. Now expressing ${}_4F_3$ in power series form and interpreting the result in the form of Srivastava function $F^{(3)}$ [12; p.428], we get

$$K_1[a,a,a,a;b,b,b,c;d,e,f,d;x,y,y,t]=F^{(3)}\left[\begin{matrix}a::b;\_;\_:\_;\dfrac{e+f}{2},\dfrac{e+f-1}{2};c;x,4y,t\\ \_::\_;\_;d:\_;e,f,e+f-1;\_;\end{matrix}\right] \tag{2.3}$$

when $t=0$ or $c=0$, (2.3) reduces to

$$F_C^3[a;b,e,f;x,y,y]=F^{2:0;2}_{0:1;3}\left[\begin{matrix}a,b:-;\dfrac{e+f}{2},\dfrac{e+f-1}{2};x,4y\\ -:d;e,f,e+f-1;\end{matrix}\right], \tag{2.4}$$

where $F_C^{(3)}$ and $F^{2:0;2}_{0:1;3}$ are Lauricella [9,p.114] and Kampé de Fériet [14;p.23 (1.2)] functions, respectively.

When $x=0$ in (2.3), we get the following correct form of the reduction formula (1.4) and (1.5) of Exton [8,p. 134 (4.7.8)], for Saran's function F_E [11].

$$F_E[a,a,a;c,b,b;d,e,f,c;t,y,y]=F^{1:3;3}_{0:3;3}\left[\begin{matrix}a:A,B,C;b,\dfrac{e+f}{2},\dfrac{e+f-1}{2};t,4y\\ \_:A,B,d;e,e+f-1;\end{matrix}\right] \tag{2.5}$$

Setting $f=e$, $z=-y$ and $t=x$ in (2.1) and using a result of Srivastava [13; p.296 (9)]. we get

$$K_1[a,a,a,a;b,b,b,c;d,e,e,d;x,y,-y,x]=\sum_{m,q=0}^{\infty}\frac{(a)_{m+q}(b)_m(c)_q x^{m+q}}{(d)_{m+q}m!q!}$$

$$4F_3\left[\begin{matrix}\frac{a+m+q}{2},\frac{a+m+q+1}{2},\frac{b+m}{2},\frac{b+m+1}{2};\\ e,\frac{e}{2},\frac{e+1}{2};\end{matrix}-4y^2\right]. \qquad ...(2.6)$$

Now expressing ${}_4F_3$ in power series form and using a result [10;p. 22(lemma 5)], we get

$$K_1[a,a,a,a;b,b,b,c;d,e,e,d;x,y,-y,x]$$

$$=\sum_{n=0}^{\infty}\frac{(a)_{2n}(b)_{2n}\left(-y^2\right)^n}{(e)_n(e)_{2n}n!}F_1[a+2n;b+2n,c;d;x,x], \qquad ...(2.7)$$

where F_1 is Appell's function of first kind [1;p. 14 (11)].

Again, using a result [5; p. 239(11) for F_1 in (2.7), and a result of Carlson [4; p. 234(10)] for the resulting Gauss function ${}_2F_1$ [10, p. 45 (1), we get the following reduction of K_1 into a combination of two Kampé de Fériet's functions.

$$K_1[a,a,a,a;b,b,b,c;d,e,e,d;x,y,-y,x]$$

$$=F^{4:0,2}_{0:3,5}\left[\begin{matrix}\frac{a}{2},\frac{a+1}{2},\frac{a+c}{2},\frac{b+c+1}{2}:-;\frac{b}{2},\frac{b+1}{2};\\ -:\frac{1}{2},\frac{d}{2},\frac{d+1}{2};e,\frac{e}{2},\frac{e+1}{2},\frac{b+c}{2},\frac{b+c+1}{2};\end{matrix}x^2,-4y^2\right]+\frac{a(b+c)x}{d}$$

$$\times F^{4:0,2}_{0:3,5}\left[\begin{matrix}\frac{a+1}{2},\frac{a+2}{2},\frac{b+c+1}{2},\frac{b+c+2}{2}:-;\frac{b}{2},\frac{b+1}{2};\\ -:\frac{3}{2},,\frac{d+1}{2};\frac{d+2}{2};e,\frac{e}{2},\frac{e+1}{2},\frac{b+c}{2},\frac{b+c+1}{2};\end{matrix}x^2,-4y^2\right]. \qquad ...(2.8)$$

when c=0 in (2.8), we get

$$F_C^{(3)}[a;b;d,e,e;x,y,-y]$$

$$=F^{4:0,0}_{0:3,3}\left[\begin{matrix}\frac{a}{2},\frac{a+1}{2},\frac{b}{2},\frac{b+1}{2}:-;-;\\ -:\frac{1}{2},\frac{d}{2},\frac{d+1}{2};e,\frac{e}{2},\frac{e+1}{2};\end{matrix}x^2,-4y^2\right]+\frac{abx}{d}$$

$$= \times F^{4:0,0}_{0:3,3}\left[\begin{matrix} \frac{a+1}{2},\frac{a+2}{2},\frac{b+1}{2},\frac{b+2}{2}:-;-; \\ -:\frac{3}{2},\frac{d+1}{2};\frac{d+2}{2};e,\frac{e}{2},\frac{e+1}{2}; \end{matrix}\; x^2,-4y^2\right]. \qquad ...(2.9)$$

3. Reducibility of K_2 Function. Writing (1.2) in the form

$$K_2[a,a,a,a;b,b,b,c;d,e,f,g;x,y,z,t]=\sum_{p,q=0}^{\infty}\frac{(a)_{p+q}(b)_p(c)_q z^p t^q}{(f)_p(g)_q\,p!q!}$$

$$F_4[a+p+q;b+p;d,e;x,y]\,. \qquad ...(3.1)$$

Putting $y=x$ in (3.1) and using [3; p. 101(37)], we get

$$K_2[a,a,a,a;b,b,b,c;d,e,f,g;x,x,z,t]$$

$$=F^{(3)}\left[\begin{matrix} a::b;-;-:\frac{d+e}{2},\frac{d+e-1}{2},-;c; \\ -::-;-;-:d,e,d+e+1,f;g; \end{matrix}\;4x,z,t\right]. \qquad ...(3.2)$$

Setting. $z=-x$ and $f=d$ in (3.1) and using a result of Srivastava [13; p. 296 (9)], we get

$$K_2[a,a,a,a;b,b,b,c;d,e,d,g;x,y,-x,t]=\sum_{n,q=0}^{\infty}\frac{(a)_{n+q}(b)_m(c)_q y^n t^q}{(e)_n(g)_q\,n!q!}$$

$$\times {}_4F_3\left[\begin{matrix} \frac{a+n+q}{2}, & \frac{a+n+q+1}{2}, & \frac{b+n}{2}, & \frac{b+n+1}{2}; \\ d\;\;, & \frac{d}{2}\;\;, & \frac{d+1}{2}; & \end{matrix}\;-4x^2\right]. \qquad ...(3.3)$$

Now expanding ${}_4F_3$ into series form and using [10; p. 22 (Lemma 5)], we get

$$K_2[a,a,a,a;b,b,b,c;d,e,d,g;x,y,-x,t]$$

$$=\sum_{m=0}^{\infty}\frac{(a)_{2m}(b)_{2m}\left(-x^2\right)^m}{(d)_m(d)_{2m}\,m!}F_2[a+2m;b+2m,c;e,g;y,t], \qquad ...(3.4)$$

where F_2 is Appell's function of second kind [1; p. 14(12)]. Putting $g=c$ in (3.4), making use of a result [5;p.238(2)] together with a result of Carlson [4;p.234 (10)], we get

$$K_2[a,a,a,a;b,b,b,c;d,e,d,c;x,y,-x,t]$$

$$= (1-t)^{-a} F_{0:3,3}^{4:0,0}\left[\begin{matrix} \frac{a}{2},\frac{a+1}{2},\frac{b}{2},\frac{b+1}{2}:-;-; \\ -:d,\frac{d}{2},\frac{d+1}{2};\frac{1}{2}e,\frac{e}{2},\frac{e+1}{2}; \end{matrix} -\left(\frac{2x}{1-t}\right)^2,\left(\frac{y}{1-t}\right)^2\right]$$

$$+\frac{aby}{e(1-t)^{a+1}} F_{0:3,3}^{4:0,0}\left[\begin{matrix} \frac{a+1}{2},\frac{a+2}{2},\frac{b+1}{2},\frac{b+2}{2}:-;-; \\ -:d,\frac{d}{2},\frac{d+1}{2};\frac{3}{2},\frac{e+1}{2},\frac{e+2}{2}; \end{matrix} -\left(\frac{2x}{1-t}\right)^2,\left(\frac{y}{1-t}\right)^2\right]. \quad ...(3.5)$$

When $y=0$, (3.5) yields a reduction F_E into ${}_4F_3$ in the form

$$F_E[a,a,a;c,b,b,c,d,d;t,x,-x] = (1-t)^{-a}{}_4F_3\left[\begin{matrix} \frac{a}{2},\frac{a+1}{2};\frac{b}{2},\frac{b+1}{2}: \\ d,\frac{d}{2},\frac{d+1}{2}; \end{matrix} \left(\frac{-4x}{1-t}\right)^2\right]. \quad ...(3.6)$$

4. Reducibility of K_3- Function. Expressing (1.3) in the form

$$K_3[a,a,a,a;b,b,c,c;d,e,e,d;x,y,z,t]$$

$$= \sum_{m,q=0}^{\infty} \frac{(a)_{m+q}(b)_m(c)_q x^m t^q}{(d)_{m+q} m!q!} F_1[a+m+q;b+m;c+q,e;y,z]. \quad ...(4.1)$$

Putting $z=y$ in (4.1) and using [5; p. 239 (11)], we get a reduction formula of K_3 into $F^{(3)}$ in the form

$$K_3[a,a,a,a;b,b,c,c;d,e,e,d;x,y,y,t] = F^{(3)}\left[\begin{matrix} a,b+c::-;-;-:b;-;c; \\ -::-;-;d,b+c:-e;-; \end{matrix} x,y,t\right]. \quad ...(4.2)$$

When $t=0$, (4.2) reduces to

$$F_F[a,a,a;b,c,b;d,e,e;x,y,y] = F_{0:2;1}^{2:1;0}\left[\begin{matrix} a,b+c;b;-; \\ -:d,b+c,:e; \end{matrix} x,y\right], \quad ...(4.3)$$

where F_F is another Saran's function[11].

Furthermore for $d=b$,(4.3) reduces to a reduction formula of F_F into F_4 in the form

$$F_F[a,a,a;b,c,b;b,e,e;x,y,y] = F_4[a;b+c;b+c,c;x,y]. \quad ...(4.4)$$

Again, writing

$$K_3[a,a,a,a;b,b,c,c;a-b-c+1,e,e,a-b-c+1,x,y,z,x]$$

$$= \sum_{n,p=0}^{\infty} \frac{(a)_{n+p}(b)_n(c)_p y^n z^p}{(e)_{n+p} n!\, p!} F_1[a+n+p; b+n; c+p, a-b-c+1; x, x], \qquad \text{...(4.5)}$$

and making use of [5, p. 239(11)] and Goursat's quadratic transformation [5, p. 113(34)], we have

$$K_3[a,a,a,a;b,b,c,c;a-b-c+1,e,e,a-b-c+1,x,y,z,x]$$

$$= (1+x)^{-a} \sum_{n,p=0}^{\infty} \frac{(a)_{n+p}(b)_n(c)_p \left(\frac{y}{1+x}\right)^n \left(\frac{z}{1+x}\right)^p}{(e)_{n+p} n!\, p!} {}_2F_1\left[\begin{matrix} \frac{a+n+p}{2}, \frac{a+n+p+1}{2}; \\ a-b-c-1; \end{matrix} \frac{4x}{(1+x)^2}\right]. \qquad (4.6)$$

Now using a double series identity of Srivastava [15; p. 196 (23)]

$$\sum_{n,p=0}^{\infty} A(n,p) = \sum_{n,p=0}^{\infty} A(2n,2p) + \sum_{n,p=0}^{\infty} A(2n+1,2p)$$

$$+ \sum_{n,p=0}^{\infty} A(2n,2p+1) + \sum_{n,p=0}^{\infty} A(2n+1,2p+1) \qquad \text{...(4.7)}$$

in (4.6), we get a reduction of K_3 in terms of combination of four $F^{(3)}$'s in the form

$$K_3[a,a,a,a;b,b,c,c;a-b-c+1,e,e,a-b-c+1;x,y,z,x]$$

$$= (1+x)^{-a} F^{(3)}\left[\begin{matrix} \frac{a}{2}, \frac{a+1}{2} :: -;-;- : -; \frac{b}{2}, \frac{b+1}{2}; \frac{c}{2}; \frac{c+1}{2}; \\ - :: -; \frac{e}{2}, \frac{e+1}{2}; - : a-b-c+1; \frac{1}{2}; \frac{1}{2}; \end{matrix} \frac{4x}{(1+x)^2}, \left(\frac{y}{1+x}\right)^2, \left(\frac{z}{1+x}\right)^2\right]$$

$$+ \frac{aby}{e(1+x)^{a+1}} F^{(3)}\left[\begin{matrix} \frac{a+1}{2}, \frac{a+2}{2} :: -;-;- : -; \frac{b+1}{2}, \frac{b+2}{2}; \frac{c}{2}; \frac{c+1}{2}; \\ - :: -; \frac{e+1}{2}, \frac{e+2}{2}; - : a-b-c+1; \frac{3}{2}; \frac{1}{2}; \end{matrix} \frac{4x}{(1+x)^2}, \left(\frac{y}{1+x}\right)^2, \left(\frac{z}{1+x}\right)^2\right]$$

$$+ \frac{acz}{e(1+x)^{a+1}} F^{(3)}\left[\begin{matrix} \frac{a+1}{2}, \frac{a+2}{2} :: -;-;- : -; \frac{b}{2}, \frac{b+1}{2}; \frac{c+1}{2}; \frac{c+2}{2}; \\ - :: -; \frac{e+1}{2}, \frac{e+2}{2}; - : a-b-c+1; \frac{1}{2}; \frac{3}{2}; \end{matrix} \frac{4x}{(1+x)^2}, \left(\frac{y}{1+x}\right)^2, \left(\frac{z}{1+x}\right)^2\right]$$

$$+ \frac{a(a+1)bcyz}{e(e+1)(1+x)^{2+a}} F^{(3)}\left[\begin{matrix} \frac{a+2}{2}, \frac{a+3}{2} :: -;-;- : -; \frac{b+1}{2}; \frac{b+2}{2}; \frac{c+1}{2}; \frac{c+2}{2}; \\ - :: -; \frac{e+2}{2}, \frac{e+3}{2}; - - : a-b-c+1; \frac{3}{2}; \frac{3}{2}; \end{matrix} \frac{4x}{(1+x)^2}, \left(\frac{y}{1+x}\right)^2, \left(\frac{z}{1+x}\right)^2\right] \qquad \text{...(4.8)}$$

When $y=0$ in (4.8), we get a reduction for Saran function F_F involving the combination of the two Kampé de Fériet's functions

$$F_F\left[a,a,a;c,b,c;a-b-c+1,a-b-c+1;z,x,x\right]$$

$$=(1+x)^{-a}F_{0:1,3}^{2:0,2}\left[\begin{matrix}\frac{a}{2},\frac{a+1}{2}:-;\frac{c}{2};\frac{c+1}{2};\\ -:a-b-c+1;\frac{1}{2};\frac{e}{2},\frac{e+1}{2}\end{matrix}\frac{4x}{(1+x)^2},\left(\frac{z}{1+x}\right)^2\right]$$

$$+\frac{acz}{e(1+x)^{a+1}}F_{0:1,3}^{2:0,2}\left[\begin{matrix}\frac{a+1}{2},\frac{a+2}{2}:-;\frac{c}{2};\frac{c+1}{2};\\ -:a-b-c+1;\frac{3}{2};\frac{e+1}{2},\frac{e+2}{2}\end{matrix}\frac{4x}{(1+x)^2},\left(\frac{z}{1+x}\right)^2\right]. \quad ...(4.9)$$

REFERENCES

[1] P. Appell, et J. Kampé de Fériet, *Fonctions Hypergeometriques et Hyperspheriques. Polinomes d'Hermite*, Gauthier-Villars, Paris, 1926.

[2] J.L. Burchnall and T.W. Chaundy, Expansions of Appell's double hypergeometric function (II), *Quart. J. Math.*, **12** (Oxford series) (1941), 112-128.

[3] J.L. Burchnall, Differential equations associated with hypergeometric functions, *Quat.* J. Math., (Oxford series), **13** (1942), 90-106.

[4] B.C. Carlson, Some extensions of Lardner's relations between $_0F_3$ and Bessel function, *SLAM J. Math. Anal.*, **1(2)** (1970), 232-242.

[5] A. Erdélyi, et al., Higher Transcendental functions, Vol.1 McGraw-Hill, New York (1953).

[6] H. Exton, Certain hypergeometric functions of four variables, *Bull. Greek Math. Soc. N.S.*, **13** (1972), 104-113.

[7] H. Exton, On the reducibility of the triple hypergeometric functions F_E, F_G and H_B, *Indian J. Math.*, **18(2)** (1976), 63-69.

[8] H. Exton, *Multiple Hypergeometric Functions and Applications*, Ellis Horwaood Limited, Chichester (1976).

[9] G. Lauicella , Sulle Funzioni Ipergeometriche a piú variabili, *Rend. Circ. Mat. Palermo*, **7** (1893), 111-158.

[10] E. D. Rainville, Special Functions, The Macmilan Comp., New York , 1960, Reprinted Chelsea Pub'. Co. Bronx., New Yrok, 1971.

[11] S. Saran, Hypergeometric functions of three variables, *Ganita*, **5** (1954), 77-91.

[12] H. M. Srivastava, Generalized Neumann expansions involving hypergeometric functions, *Proc. Camb. Philos. Soc.*, **63(1-2)** (1967), 425-429.

[13] H.M. Srivastava, On the reducibility of Appell's function F_4, *Canad. Math. Bull.*, **16(2)** (1973), 295-298.

[14] H.M. Srivastava and M.A. Pathan, Some bilateral generating functions for the extended jacobi polynomials-I, *Comment. Math. Univ. St. Pauli.*, **28(1)** (1979), 23-30.

[15] H.M. Srivastava, A note on certain identities involving generalized hypergeometric Series, *Nederal. Akad. Wetersch. Proc. Ser. A.* **82(2)**=*Indag. Math.*, **41(2)** (1979), 191-201.

Jñānābha, Vol. 38, 2008

CONVERGENCE RESULT OF (L,α) UNIFORM LIPSCHITZ ASYMPTOTICALLY QUASI NONEXPANSIVE MAPPINGS IN UNIFORMLY CONVEX BANACH SPACE

By

G.S. Saluja

Department of Mathematics and I.T.

Government College of Science, Raipur-4920101, Chhattisgarh

E-mail : saluja_1963@rediffmail.com

(Received : February 25, 2008)

ABSTRACT

In this paper, we have proved common fixed point of the modified Ishikawa iterative sequences with errors for two (L,α) uniform Lipschitz asymptotically quasi nonexpansive mappings in a unifromly convex Banach space. Our result extends and improves the corresponding result of Khan and Takahashi [1], Tan and *Xu* [5] and many others.

2000 Mathematics Subject Classification : 47H10.

Keywords : Asymptotically quasi nonexpansive mapping, common fixed point, (L,α) uniform Lipschitz mapping, the modified Ishikawa iteration scheme with errors, uniformly convex Banach space.

1. Introduction and Preliminaries. Let C be a subset of a normed space E, and let T be a self mapping of C. The mapping T is said to be an asymptotically nonexpansive mapping, if there exists a sequence $\{k_n\}$ in $[0,\infty)$ with $\lim_{n\to\infty} k_n = 0$ such that

$$\|T^n x - T^n y\| \le (1+k_n)\|x-y\|$$

for all $x, y \in C$ and $n \ge 1$.

The mapping T is said to be an asymptotically quasi nonexpansive mapping, if there exists a sequence $\{k_n\}$ in $[0,\infty)$ with $\lim_{n\to\infty} k_n = 0$ and $F(T)\ne\phi$ such that

$$\|T^n x - p\| \le (1+k_n)\|x-p\|$$

for all $x, y \in C$, for all $p \in F(T)$ and $n \ge 1$.

It there are constant $L>0$ and $\alpha>0$, such that

$$\|T^n x - T^n y\| \le L\|x-y\|^{\alpha}$$

for all $x,y \in C$ and $n \in N$, then T is (L,α) uniform Lipschitz.

In 1994, Tan and *Xu* [5] had proved the problem on convergence of Ishikawa iteration for asymtotically nonexpansive mapping on a compact convex subset of a uniformly convex Banach space. In 2001, Qihou [2], presents the necessary and sufficient conditions for the Ishikawa iteration of asymptotically quasi-nonexpansive mapping with an error member on a Banach space converging to a fixed point. Again in 2002, the same author has proved the convergence of Ishikawa iteration of a (L,α) unifrom Lischitz asymptotically quasi nonexpansive mapping with an error member on a compact convex subset of a uniformly convex Banach space based on some results of [2].

Recently, Khan and Takahashi [1] considered the problems of approximating common fixed point of two asymptotically nonexpansive self mappings S and T of C through weak and strong convergence of the iterative sequence $\{x_n\}$ defined by

$$x_1 \in C$$

$$x_{n+1} = (1-a_n)x_n + a_n S^n y_n, \quad n \geq 1$$

$$y_n = (1-b_n)x_n + b_n T^n x_n, \quad n \geq 1 \tag{A}$$

where $\{a_n\}$ and $\{b_n\}$ are some sequences in $[0,1]$.

Motivated and inspired by Khan and Takahashi [1] and others we study the following iteration scheme: Let C be a nonempty subset of a Banach space E and $S,T:C \to C$ be two asymptotically quasi nonexpansive mappings. Consider the following iterative sequence $\{x_n\}$ with errors defined by

$$x_0 \in C$$

$$x_{n+1} = a_n x_n + b_n S^n y_n + c_n u_n,$$

$$y_n = \bar{a}_n x_n + \bar{b}_n T^n x_n + \bar{c}_n v_n, \tag{B}$$

where $u_n, v_n \in C$, $\forall n \in C, 0 \leq a_n, b_n, c_n \bar{a}_n, \bar{b}_n, \bar{c}_n \leq 1$, $a_n + b_n + c_n = 1 = \bar{a}_n + \bar{b}_n + \bar{c}_n$, for all $n \in N, \sum_{n=1}^{\infty} c_n < +\infty, \sum_{n=1}^{\infty} \bar{c}_n < +\infty$.

In this paper, we have proved convergence of common fixed point of modified Ishikawa iteration scheme with errors of (L,α) uniform Lipschitz asymptotically quasi nonexpansive mappings on a compact convex subset of a uniformly convex Banach space. Our result extends and improves the corresponding result of Tan and *Xu* [5], Khan and Takahashi [1] and many others.

We need to following result and lemmas to prove our main result:

Theorem LQ [2, Theorem 3]: Let E be a nonempty closed convex subset of a

Banach space, T is an asymptotically quasi-nonexpansive mapping on E, and $F(T)$ nonempty. Given $\sum_{n=1}^{\infty} u_n < +\infty$, $\forall x_1 \in E$, defined $\{x_n\}_{n=1}^{\infty}$ as

$$x_{n+1} = a_n x_n + b_n T^n y_n + c_n m_n,$$

$$y_n = \bar{a}_n x_n + \bar{b}_n T^n x_n + \bar{c}_n l_n, \forall n \in N,$$

where m_n, $l_n \in E$, and $\{\|m_n\|\}_{n=1}^{\infty}$, $\{\|l_n\|\}_{n=1}^{\infty}$ are bounded, $a_n + b_n + c_n = 1 = \bar{a}_n + \bar{b}_n + \bar{c}_n, 0 \le a_n, b_n, \bar{c}_n, \bar{a}_n, \bar{b}_n, \bar{c}_n \le 1$. Then $\{x_n\}_{n=1}^{\infty}$ converges to some fixed point p of T if and only if there exists some infinite subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ which converges to p.

Lemma 1.1 [J.Schu's Lemma]: Let X be a uniformly convex Banach space, $0 < \alpha \le t_n \le \beta < 1$, $x_n, y_n \in X$, $\limsup_{n\to\infty}\|x_n\| \le a$, $\limsup_{n\to\infty}\|y_n\| \le a$, and $\lim_{n\to\infty}\|t_n x_n + (1-t_n) y_n\| = a$, $a \ge 0$. Then $\lim_{n\to\infty}\|x_n - y_n\| = 0$.

Lemma 1.2 [2, Lemma 2] : Let nonnegative series $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{\gamma_n\}_{n=1}^{\infty}$ satisfy $\alpha_n \le (1+\beta_n)\alpha_n + \gamma_n$, $\forall n \in N$, and $\sum_{n=1}^{\infty} \beta_n < +\infty$, $\sum_{n=1}^{\infty} \gamma_n < +\infty$; then $\lim_{n\to\infty} \alpha_n$ exits.

Lemma 1.3: Let C be a nonempty convex subset of normed space E, $S,T: C \to C$ are two asymptotically quasi nonexpansive mappings, and $F = F(S) \cap F(T)$ nonempty, $\sum_{n=1}^{\infty} k_n < +\infty$, for all $x_0 \in C$, let

$$x_{n+1} = a_n x_n + b_n S^n y_n + c_n u_n,$$

$$y_n = \bar{a}_n x_n + \bar{b}_n T^n x_n + \bar{c}_n v_n,$$

where $u_n, v_n \in C, \forall n \in C, 0 \le a_n, b_n, c_n, \bar{a}_n, \bar{b}_n, \bar{c}_n \le 1$, $a_n + b_n + c_n = 1 = \bar{a}_n + \bar{b}_n + \bar{c}_n$, for all $n \in N, \sum_{n=1}^{\infty} c_n < +\infty, \sum_{n=1}^{\infty} \bar{c}_n < +\infty$. Then

(a) $\|x_{n+1} - p\| \le (1+k_n)^2 \|x_n - p\| + m_n, \forall n \in N, \forall p \in F = F(S) \cap F(T)$,

where $m_n = b_n(1+k_n)\bar{c}_n\|v_n - p\| + c_n\|u_n - p\|$.

(b) There exists a constant $M>0$ such that $\|x_{n+m} - p\| \le M\|x_n - p\|$

$+ M\sum_{k=n}^{n+m-1} m_k, \forall n, m \in N, \forall p \in F = F(S)\cap F(T)$.

Proof of (a) For any $p\in F$, we have from (B)

$$\|x_{n+1}-p\| \le a_n\|x_n-p\|+b_n\|S^n y_n-p\|+c_n\|u_n-p\|$$

$$\le a_n\|x_n-p\|+b_n(1+k_n)\|y_n-p\|+c_n\|u_n-p\| \quad (1)$$

and

$$\|y_n-p\| \le \bar{a}_n\|x_n-p\|+\bar{b}_n\|T^n x_n-p\|+\bar{c}_n\|u_n-p\|$$

$$\le \bar{a}_n\|x_n-p\|+\bar{b}_n(1+k_n)\|x_n-p\|+\bar{c}_n\|v_n-p\|$$

$$\le \{\bar{a}_n+\bar{b}_n(1+k_n)\}\|x_n-p\|+\bar{c}_n\|v_n-p\| \quad (2)$$

From (1) and (2), we have

$$\|x_{n+1}-p\| \le a_n\|x_n-p\|+b_n(1+k_n)\left[\{\bar{a}_n+\bar{b}_n(1+k_n)\}\|x_n-p\|+\bar{c}_n\|v_n-p\|\right]+c_n\|u_n-p\|$$

$$\le a_n\|x_n-p\|+b_n(1+k_n)\bar{a}_n\|x_n-p\|+b_n\bar{b}_n(1+k_n)^2\|x_n-p\|$$

$$+b_n\bar{c}_n(1+k_n)\|v_n-p\|+c_n\|u_n-p\|$$

$$\le a_n\|x_n-p\|+(1-a_n-c_n)(1+k_n)\bar{a}_n\|x_n-p\|+(1-a_n-c_n)$$

$$\bar{b}_n(1+k_n)^2\|x_n-p\|+m_n$$

$$\le a_n\|x_n-p\|+(1-a_n)(1+k_n)\bar{a}_n\|x_n-p\|+(1-a_n)\bar{b}_n(1+k_n)^2\|x_n-p\|+m_n$$

$$\le a_n\|x_n-p\|+(1-a_n)(1+k_n)^2\bar{a}_n\|x_n-p\|+(1-a_n)\bar{b}_n(1+k_n)^2\|x_n-p\|+m_n$$

$$\le a_n(1+k_n)^2\|x_n-p\|+(1-a_n)(1+k_n)^2(\bar{a}_n+\bar{b}_n)\|x_n-p\|+m_n$$

$$\le a_n(1+k_n)^2\|x_n-p\|+(1-a_n)(1+k_n)^2\|x_n-p\|+m_n$$

$$\le (1+k_n)^2\|x_n-p\|+m_n$$

where $m_n = b_n\bar{c}_n(1+k_n)\|v_n-p\|+c_n\|u_n-p\|$.

Proof of (b) Since $1+x \leq e^x$ for all $x>0$. Therefore from (a) it can be obtained that

$$\begin{aligned}
\|x_{n+m}-p\| &\leq (1+u_{n+m-1})^2 \|x_{n+m-1}-p\| + m_{n+m-1} \\
&\leq e^{2u_{n+m-1}} \|x_{n+m-1}-p\| + m_{n+m-1} \\
&\leq e^{2(u_{n+m-1}+u_{n+m-2})} \|x_{n+m-2}-p\| + e^{2u_{n+m-1}m_{n+m-2}+m_{n+m-1}} \\
&\leq e^{2(u_{n+m-1}+u_{n+m-2})} \|x_{n+m-2}-p\| + e^{2u_{n+m-1}}(m_{n+m-2}+m_{n+m-1}) \\
&\leq \quad \dots \quad \dots \quad \dots \\
&\leq e^{2\sum_{k=n}^{n+m-1} u_k} \|x_n-p\| + e^{2\sum_{k=n}^{n+m-1} u_k} \sum_{k=n}^{n+m-1} m_k \\
&\leq M\|x_n-p\| + m \sum_{k=n}^{n+m-1} m_k
\end{aligned}$$

where $M = e^{2\sum_{k=n}^{\prime} u_k}$.

This completes the proof of (**b**).

2. Main Result.

Theorem 2.1 : Let C be a nonempty compact subset of uniformly convex Banach space X, and $S,T: C \to C$ be two (L,α) uniform Lipschitz asymptotically quasi nonexpansive mappings with sequence $\{k_n\} \in [0,\infty)$ such that $\sum_{n=1}^{\infty} k_n < +\infty$. Let $\forall_{x_0} \in C$, and

$$x_{n+1} = a_n x_n + b_n S^n y_n + c_n u_n,$$

$$y_n = \bar{a}_n x_n + \bar{b}_n T^n x_n + \bar{c}_n v_n,$$

where $u_n, v_n \in C, \forall n \in C, 0 \leq a_n, b_n, c_n, \bar{a}_n, \bar{b}_n, \bar{c}_n \leq 1$, $a_n + b_n + c_n = \bar{a}_n + \bar{b}_n + \bar{c}_n = 1$,

$0 < \alpha \leq a_n \leq \bar{a} < 1, 0 < \alpha \leq \bar{a}_n, 0 < \beta \leq b_n \leq \bar{\beta} < 1, \bar{b}_n \leq \beta < 1, \forall n \in N$, $\lim_{n\to\infty} \bar{b}_n = 0$,

$\sum_{n=1}^{\infty} c_n < +\infty$, $\sum_{n=1}^{\infty} \bar{c}_n < +\infty$. If $F = F(S) \cap F(T) \neq \phi$. Then $\{x_n\}_{n=1}^{\infty}$ converges to some common fixed point p of S and T.

Proof. For any $p \in F$, we have from Lemma 1.3(a)

$$\|x_{n+1} - p\| \le (1 + k_n)^2 \|x_n - p\| + m_n,$$

where $m_n = b_n \bar{c}_n (1 + k_n) \|v_n - p\| + c_n \|u_n - p\|$. Since $\sum_{n=1}^{\infty} k_n < +\infty$, $\sum_{n=1}^{\infty} c_n < +\infty$, $\sum_{n=1}^{\infty} \bar{c}_n < +\infty$, C is bounded, thus we know from Lemma 1.2 that $\lim_{n\to\infty} \|x_n - p\|$ exists.

On the other hand, we have

$$\begin{aligned} \|y_n - p\| &\le \bar{a}_n \|x_n - p\| + \bar{b}_n \left\|T^n x_n - p\right\| + \bar{c}_n \|v_n - p\| \\ &\le \bar{a}_n \|x_n - p\| + \bar{b}_n (1 + k_n) \|x_n - p\| + \bar{c}_n \|v_n - p\| \end{aligned}$$

Thus

$$\limsup_{n\to\infty} \|y_n - p\| \le \limsup_{n\to\infty} \|x_n - p\| = \lim_{n\to\infty} \|x_n - p\|.$$

Note that

$$\limsup_{n\to\infty} \left\|S^n y_n - p\right\| \le \limsup_{n\to\infty} (1 + k_n) \|y_n - p\| \le \lim_{n\to\infty} \|x_n - p\|.$$

and

$$\begin{aligned} \lim_{n\to\infty} \|x_{n+} - p\| &= \lim_{n\to\infty} \left\|a_n x_n + b_n S^n y_n + c_n u_n - p\right\| \\ &= \lim_{n\to\infty} \left\| a_n \left[x_n - p \frac{c_n}{2a_n} (u_n - p) \right] + b_n \left[S^n y_n - p + \frac{c_m}{2b_n} (u_n - p) \right] \right\| \\ &= \lim_{n\to\infty} \|x_n - p\|. \end{aligned}$$

Thus from J. Schu's Lemma, we have

$$\lim_{n\to\infty} \left\| x_n - S^n y_n + \left(\frac{c_n}{2a_n} - \frac{c_n}{2b_n} \right) (u_n - p) \right\| = 0.$$

Note that $\lim_{n\to\infty} \left\| \left(\frac{c_n}{2a_n} - \frac{c_n}{2b_n} \right) (u_n - p) \right\| = 0.$

Therefore we have

$$\lim_{n\to\infty} \left\| x_n - S^n y_n \right\| = 0 \quad . \tag{3}$$

Since C is compact, the sequence $\{x_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$.

Let

$$\lim_{k\to\infty} x_{n_k} = p. \qquad ...(4)$$

Thus from (3) and $\lim_{n\to\infty} c_n = 0$, we have

$$\begin{aligned} \left\|x_{n_k+1} - x_{n_k}\right\| &= \left\|a_{n_k}x_{n_k} + b_{n_k}S^{n_k}y_{n_k} + c_{n_k}u_{n_k} - x_{n_k}\right\| \\ &= \left\|\left(1-b_{n_k}-c_{n_k}\right)x_{n_k} + b_{n_k}S^{n_k}y_{n_k} + c_{n_k}u_{n_k} - x_{n_k}\right\| \\ &\le b_{n_k}\left\|S^{n_k}y_{n_k} - x_{n_k}\right\| + c_{n_k}\left\|u_{n_k} - x_{n_k}\right\| \to 0 \text{ as } k\to\infty. \end{aligned} \qquad (5)$$

Note that

$\lim_{n\to\infty} \bar{b}_n = 0, \lim_{n\to\infty} \bar{c}_n = 0;$

therefore, we have

$$\begin{aligned} \left\|y_n - x_n\right\| &= \left\|\bar{a}_n x_n + \bar{b}_n T^n x_n + \bar{c}_n v_n - x_n\right\| \\ &= \left\|\left(1-\bar{b}_n-\bar{c}_n\right)x_n + \bar{b}_n T^n x_n + \bar{c}_n v_n - x_n\right\| \\ &\le \bar{b}_n\left\|T^n x_n - x_n\right\| + \bar{c}_n\left\|v_n - x_n\right\| \to 0 \text{ as } n\to\infty \end{aligned} \qquad (6)$$

Thus from (3) and (4), we have

$$\lim_{k\to\infty} S^{n_k}y_{n_k} = p. \qquad (7)$$

Thus $\lim_{k\to\infty} x_{n_k+1} = p$. Similary, $\lim_{k\to\infty} x_{n_k+2} = p$ and

$$\lim_{k\to\infty} S^{n_k+1}y_{n_k+1} = p. \qquad (8)$$

Now

$$\left\|x_n - T^n y_n\right\| \le \left\|y_n - x_n\right\| + \left\|y_n - T^n y_n\right\| \to 0 \text{ as } n\to\infty. \qquad (9)$$

Therefore from (4) and (9), we have

$$\lim_{k\to\infty} T^{n_k}y_{n_k} = p. \qquad (10)$$

Thus $\lim_{k\to\infty} x_{n_k+1} = p$ and since $\lim_{k\to\infty} x_{n_k+2} = p$ so

$$\lim_{k\to\infty} T^{n_k+1} y_{n_k+1} = p \tag{11}$$

From (3) – (8), we have

$$\|p - Sp\| \le \left\|p - S^{n_k+1} y_{n_k+1}\right\| + \left\|S^{n_k+1} y_{n_k+1} - S^{n_k+1} x_{n_k+1}\right\| + \left\|S^{n_k+1} x_{n_k+1} - S^{n_k+1} x_{n_k}\right\|$$

$$+ \left\|S^{n_k+1} x_{n_k} - S^{n_k+1} y_{n_k}\right\| + \left\|S^{n_k+1} y_{n_k} - S_P\right\|$$

$$\le \left\|p - S^{n_k+1} y_{n_k+1}\right\| + L\left\|y_{n_k+1} - x_{n_k+1}\right\|^{\alpha} + L\left\|x_{n_k+1} - x_{n_k}\right\|^{\alpha} + L\left\|x_{n_k} - y_{n_k}\right\|^{\alpha}$$

$$+ L\left\|S^{n_k} y_{n_k} - p\right\|^{\alpha} \to 0 \text{ as } k \to \infty$$

Similarly we can show that

$$\|p - Tp\| \to 0 \text{ as } k \to \infty.$$

Thus p is a common fixed point of S and T. Since the subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ converges to p from (4), we have $\lim_{n\to\infty} x_n = p \in F = F(S) \cap F(T)$ from Theorem *LQ*. This completes the proof.

Remark 2.2 . Theorem 2.1 improves and extends the corresponding result of Khan and Takahashi [1] in several aspects.

Remark 2.3 . Theorem 2.1 also extends the corresponding result of *Tan* and *Xu* [5] to the case of more general class of asymptotically nonexpansive mappings.

Remark 2.4 . Theorem 2.1 also generalizes the result of Qihou [3].

REFERENCES

[1] S.H. Khan and W. Takahashi, Approximating common fixed points of two asymptotically nonexpansive mappings, *Sci. Math. Jpn.*, **53** (2001), 143-148.

[2] L. Qihou, Iterative sequences for asymptotically quasi-nonexpansive mapping with error member, *J. Math. Anal. Appl.*, **259** (2001), 18-24.

[3] L. Qihou, Iterative sequences for asymptotically quasi-nonexpansive mapping with an error member of uniformly covex Banach space, *J. Math. Anal. Appl.* **266** (2002), 468-471.

[4] J. Schu, Iterative construction of fixed points of strictly quasicontractive mappings, *Appl. Anal.* **40** (1991), 67-72.

[5] K.K. Tan and H.K. Xu, Fixed point iteration processes for asymptotically non-expansive mappings, *Proc. Amer. Math. Sco.*, **122** (1994), 733-739.

Jñānābha, Vol. 38, 2008

ON SPECIAL UNION AND HYPERASYMPTOTIC CURVES OF A KAEHLERIAN HYPERSURFACE

By

A. K. Singh and A.K. Gahlot

Department of Mathematics, H.N.B. Garhwal University

Campus Badshahi Thaul, Tehri Garhwal-249199, Uttarakhand

(Received : December 25, 2007, September 2, 2008)

ABSTRACT

Special, union and hypersymptotic curves of a Riemannian hypersurface have been studied by Singh [3]. The object of this paper is to investigate these curves in a Kaehlerian hypersurface.

2000 Mathematics Subject Classification : Primary 55H38; Secondary 58H32

Keywords: Hypersymptotic curves, Riemannian hypersurface, Kaehlerian hypersurface.

1. Introduction. In an $(n+1)$ dimensional real space R_{n+1} referred to an allowable co-ordinate system,

$$x^{a} \equiv \left(x^{1}, x^{2}, \ldots, x^{n+1}\right),$$

Let us introduce the metric defined by the positive Hermitian form

$$ds^{2} = 2g_{\alpha\beta}dx^{\alpha}dx^{\beta}. \qquad \text{...(1.1)}$$

If the tensor $g_{\alpha\beta}$ also satisfies the Kaehler's condition

$$\frac{\partial g_{\alpha\beta}}{\partial x^{k}} = \frac{\partial g_{ak}}{\partial x^{\beta}}, \qquad \text{...(1.2)}$$

then the space with metric satisfying the condition (1.2) is called Kaehler space. We always assume the self adjointness of the indices [2].

The angle θ between two self adjoint vector $\underline{u}$ and $\underline{v}$, whose components are S^{α} and S^{α} is given by

$$\cos\theta = \frac{g_{\alpha\beta}r^{\alpha}S^{\beta}}{RS}. \qquad \text{...(1.3)}$$

Let us consider an analytic hypersurface K_n of K_{n+1}.

If $u^{i} \equiv (u^{1}, u^{2}, \ldots, u^{n})$ denote the co-ordinates of a point in K_n, the equations of the analytic hypersurface K_n may be written in the form

$$x^{\alpha} = x^{\alpha}\left(u^{i}\right) \qquad \text{...(1.4)}$$

We qnote below some fundamental formulae form [4] which will be used in the later part of this paper. Suppose that g_{ij} is the fundamental metric tensor of K_n, then we have

$$g_{ij} = g_{\alpha\beta} B_i^\alpha B_j^\beta, \qquad \ldots(1.5)$$

where $\quad B_i^\alpha = \dfrac{\partial x^\alpha}{\partial u^i}, B_j^\beta = \dfrac{\partial x^\beta}{\partial u^j}$

If N^α be the component of unit normal vector to the hypersurface, then

$$2g_{\alpha\beta} N^\alpha N^\beta = 1 \qquad \ldots(1.6)$$

and $\quad g_{\alpha\beta} N^\alpha B_j^\beta = 0; \qquad g_{\alpha\beta} N^\beta B_j^\alpha = 0 \qquad \ldots(1.7)$

The uni vector ξ^α orthogonal to $\dfrac{dx^\alpha}{ds}$ is given by

$$g_{\alpha\beta} \frac{dx^\alpha}{ds} \xi^\beta = 0 \qquad \ldots(1.8)$$

and $\quad 2g_{\alpha\beta} \xi^\alpha \xi^\beta = 1. \qquad \ldots(1.9)$

Consider a curve $C : x^\alpha = x^\alpha(s)$ of K_n.

The components $\dfrac{dx^\alpha}{ds}$ and $\dfrac{du^i}{ds}$ of the unit tangent vector of real space, w.r.t. the nveloping space and the hypersurface are related by

$$\frac{dx^\alpha}{ds} = B_i^\alpha \frac{du^i}{ds} \qquad \ldots(1.10)$$

If q^α and P^i are the components of the first curvature vector w.r.t. K_{n+1} and K_n respectively, then

$$q^\alpha = p^i B_i^\alpha + K_n N^\alpha, \qquad \ldots(1.11)$$

where K_n is the component of the normal curvature of K_n in the direction of the curve C and

$$q^\alpha = \frac{dx^\alpha}{ds}{,}_\gamma \frac{du^j}{ds} + \frac{dx^\alpha}{ds}; \; p^i = \frac{du^i}{ds}{,}_\gamma \frac{du^j}{ds} + \frac{du^i}{ds} \qquad \ldots(1.12)$$

$$B_i^\alpha = \Omega_{ij} N^\alpha, K_n = \Omega_{ij} \frac{du^i}{ds} \frac{du^j}{ds}$$

Ω_{ij} are the components of the second fundamental tensors of the hypersurface, q^α and p^i are the components of first curvature vector of the curve with respect K_{n+1} and K_n respectively

2. Special and Union Curves in K_n. Let an n-dimensional hypersurface K_n given by equation

$$y^\alpha = y^\alpha\left(x^i\right), (\alpha = 1,2...,n+1, i = 1,2,...,n)$$

be immersed in a Kaehlerian space K_{n+1}.

The first two Frenet's formulae of a curve $x^i = x^i\ (S)$ are given by

$$\frac{\delta n^\alpha_{(0)}}{\delta s} = k_{(1)}\eta^\alpha_{(1)} \qquad \text{and} \qquad \frac{\delta n^\alpha_{(1)}}{\delta s} = k_{(1)}\eta^\alpha_{(0)} + k_{(2)}\eta^\alpha_{(2)}, \qquad ...(2.1)$$

where $\eta^\alpha_{(0)}\left(\equiv \frac{dy^\alpha}{ds}\right), \eta^\alpha_{(1)}, \eta^\alpha_{(2)}$ are components of the unit tangent, principal normal and first binormal vector and $k_{(1)}$ and $k_{(2)}$ are the first and second curvatures of the curves. The components q^α and p^i of the first curvature vector w.r.t. K_{n+1} and K_n are related by

$$q^\alpha = p^i B^\alpha_i + K_n N^\alpha, \qquad ...(2.2)$$

where, $B^\alpha_i = \frac{dy^\alpha}{dx^i}$, N^α are the components of unit normal vector and K_n is the normal curvature of the hypersurface in the direction of the curve.

Consider two congruence of the curve given by the unit ventor field $\underline{\lambda}$ and $\underline{\mu}$ which are such that the point of K_n, we have

$$\underline{\lambda}^\alpha = r^i B^\alpha_i + CN^\alpha \qquad ...(2.3a)$$

and

$$\underline{\mu}^\alpha = s^i B^\alpha_i + DN^\alpha. \qquad ...(2.3b)$$

The special $K\underline{\lambda}$ and union curves relative to $\underline{\lambda}$ have been defined by Tsagas [5] and Springer [4] respectively.

Let Γ be a $K\underline{\lambda}$ curve relative to congruence $\underline{\lambda}$ and a union curve relative to the congruence $\underline{\mu}$, then

$$\underline{\lambda}^\alpha = up^i B^\alpha_i + \omega q^\alpha, \qquad ...(2.4a)$$

and $$\underline{\mu}^{\alpha} = s\frac{dy^{\alpha}}{ds} + yq^{\alpha}, \qquad \text{...(2.4b)}$$

Using (2.2), (2.3) and (2.4), we get

$$r^i = (v+w)p^i, \qquad C = wK_n; \qquad \text{...(2.5)}$$

$$s^i = x\frac{dx^i}{ds} + yp^i, \qquad D = yK_n \qquad \text{...(2.6)}$$

Defining $R^2 = g_{ij}r^i r^j$

$$S^2 = g_{ij}s^i s^j, \quad k_{(1)}^2 = g_{ij}p^i p^j,$$

$$\cos\theta = \frac{g_{ij}r^i s^i}{RS}$$

and using equation (2.5) and (2.6), we have

$K_n S\cos\theta = DK_{(1)}$. This equation and the facts

$$1 = g_{\alpha\beta}\mu^{\alpha}\mu^{\beta} = S^2 + D^2, \qquad \text{...(2.7a)}$$

$$K_{(1)}^2 = k_{(1)}^2 + K_n^2 \qquad \text{...(2.7)b}$$

yields $$K_n = \frac{eK_{(1)}\left(1-S^2\right)^{1/2}}{\left(1-S^2\sin^2\theta\right)^{1/2}} \text{ and } K_{(1)} = \frac{eK_{(1)}S\cos\theta}{\left(1-S^2\sin^2\theta\right)^{1/2}} \qquad \text{...(2.8)}$$

where, e=1 or -1 in order that $e\cos\theta$ be non-negative. Since S and $\cos\theta$ depend upon the congruence $\underline{\lambda}$ and $\underline{\mu}$, we have the following propositions from equation (2.8):

Theorem 2.1. If a special curve relative to a fixed congruence $\underline{\lambda}$ is an union curve relative to another fixed congruence $\underline{\mu}$, then the normal and first curvature (w.r.t. K_n) at a given point of the curve are proportional to its first curvature (w.r.t. enveloping space).

Theorem 2.2. If the component of the vector fields $\underline{\lambda}$ and $\underline{\mu}$, tangent to the hypersurface are in the same direction, then the ratio of the two first curvature is equal to magnitude of the tangential (to the hypersurface) component $\underline{\mu}$.

Proof. Since r^i and s^i are along the same direction $\cos\theta = 1$ and $\sin\theta = 0$. Therefore, from (2.8), we have

$$(k_{(1)}/K_{(1)}) = S.$$

3. Hyperasymptotic Curves A hyperasymptotic curve (of order one) related to $\underline{\underline{\mu}}$ is characterized by Mishra ([1], [3])

$$\mu^\alpha = x_{(1)}\eta^\alpha_{(0)} + y_{(1)}\eta^\alpha_{(2)}. \qquad ...(3.1)$$

From the first two Frenet's formulae (refer equation(2.1)), we deduce

$$\frac{\delta q^\alpha}{\delta s} = -K^2_{(1)}\eta^\alpha_{(0)} + \left\{\frac{d}{ds}\left(\log K_{(1)}\right)\right\}q^\alpha + K_{(1)}K_{(2)}\eta^\alpha_{(2)}\,. \qquad ...(3.2)$$

Another expression for $\frac{\delta q^\alpha}{\delta s}$ can be obtained with the help of (2.2) and the relations (Mishra [2]).

$$\frac{\delta B^\alpha_i}{\delta s} = \Omega_{ij}\left(\frac{dx^i}{ds}\right)N^\alpha \quad \text{and} \quad \frac{\delta N^\alpha}{\delta s} = -\Omega_{jk}g^{ij}B^\alpha_i\frac{dx^k}{ds}$$

This later expression for $\frac{\delta q^\alpha}{\delta s}$ and (3.1), (3.2) may be used in the elimination of $\eta^\alpha_{(o)}$. In view of these equations and equations (2.2), (2.3b), we get

$$s^i = x_{(1)}\frac{dx^i}{ds} + Z\left[\frac{\delta p^i}{\delta s} - K_n\Omega_{jk}g^{ji}\frac{dx^k}{ds} - p^i\frac{d}{ds}\left(\log K_{(1)}\right) + K^2_{(1)}\frac{dx^i}{ds}\right] \qquad ,...(3.3)$$

$$D = Z\left[\Omega_{ij}p^i\frac{dx^j}{ds} + \frac{d}{ds}K_n - K_n\frac{d}{ds}\left(\log K_{(1)}\right)\right], \qquad ...(3.4)$$

where $Z = \dfrac{y_{(1)}}{K_{(1)}K_{(2)}}$.

Let $\xi^i_{(0)}, \xi^i_{(1)}, \xi^i_{(2)}$ be the unit tangent, unit principal normal, unit first binormal vectors and $k_{(1)}$, $k_{(2)}$ be the first and second curvature of the curve w.r.t. the hypersurface. The first two Frenet's formulae w.r.t. K_n yield

$$\frac{\delta p^i}{\delta s} = -k^2_{(1)}\frac{dx^i}{ds} + p^i\frac{d}{ds}\left\{\log\left(k_{(1)}\right)\right\} + k_{(1)}k_{(2)}\xi^i_{(2)}, \qquad ...(3.5)$$

From equations (2.7b), (3.3), (3.4), (3.5) and the definition

$$\cos\phi = \frac{g_{ij} s^i \left(dx^j / ds\right)}{S},$$

we deduce $x_{(1)} = S\cos\phi$ and

$$\left[\Omega_{kj} p^k \frac{dx^j}{ds} + \frac{d}{ds} K_n - K_n \frac{d}{ds}\{\log(K_{(1)})\}\right] \cdot \left(\dot{s}^i - S\cos\phi \frac{dx^i}{ds}\right)$$

$$= D\left[K_n^2 \frac{dx^i}{ds} + p^i \frac{d}{ds}\left\{\log\left(\frac{k_{(1)}}{K_{(1)}}\right)\right\} + k_{(1)} k_{(2)} \xi_{(2)}^i - K_n \Omega_{jk} g^{ij} \frac{dx^k}{ds}\right]. \quad \ldots(3.6)$$

Theorem 3.1. A hyperasymptotic curve relative to $\underline{\mu}$ is characterized by equation (3.6).

proof : Multiplying (3.6) by $g_{il} p^l$ and simplifying, we have

$$\Omega_{ij} p^i \frac{dx^j}{ds}\left(S\cos\phi + DK_n / k_{(1)}\right) = \left[Dk_{(1)} \frac{d}{ds}\left\{\log\left(\frac{k_{(1)}}{K_{(1)}}\right)\right\} - K_n S\cos\phi \frac{d}{ds}\left\{\log\left(\frac{K_n}{K_{(1)}}\right)\right\}\right],$$

where we have defined $\cos\phi = \dfrac{g_{ij} p^i s^j}{k_{(1)} S}$.

4. Hyperasymptotic and Special Curves. Let a hyperasymptotic curve w.r.t. $\underline{\mu}$ be a special curve related to $\underline{\lambda}$. This implies that $\phi = \theta$. Denoting $X = \dfrac{k_{(1)}}{K_{(1)}}$ and $Y = K_n / K_{(1)}$, writing $p^i = k_1 \xi_{(1)}^i$ and ddifferentiating the well known identity $X^2 + Y^2 = 1$, we get

$$D\frac{dX}{ds} - S\cos\theta . \frac{dY}{ds} = \Omega_{ij} \xi_{(1)}^i \left(\frac{ds^j}{ds}\right) . (XS\cos\theta + DY), \quad \ldots(4.1)$$

$$X\frac{dX}{ds} + Y\frac{dY}{ds} = 0 \quad \ldots(4.2)$$

We shall discuss the solution of the above simultaneous equations in the

following cases:

Case 1. The vector $\xi^i_{(1)}$ is not conjugate w.r.t. $\frac{dx^i}{ds}\left(i.e., \Omega_{ij}\xi^i_{(1)}\frac{ds^j}{ds} \neq 0\right)$ and the matrix

$$A = \begin{pmatrix} D & -S\cos\theta \\ X & y \end{pmatrix}$$

is non-singular. Equation (4.1) and (4.2) yield

$$\frac{dX}{ds} = \left(\Omega_{ij}\xi^i_{(1)}\frac{ds^j}{ds}\right)Y; \frac{dY}{ds} = -\left(\Omega_{ij}\xi^i_{(1)}\frac{ds^j}{ds}\right)X.$$

The quantities $X = \frac{k_{(1)}}{K_{(1)}}$ and $Y = \underset{n}{K}/K_{(1)}$ are obtained as solution of the above simultaneous equations.

Case 2. Let $\xi^i_{(1)}$ be conjugate w.r.t. $\frac{dx^i}{ds}$ and the matrix A be non-singular. Equation (4.1) and (4.2) have a unique solution given by the following

Theorem 4.1. If a special curve related to $\underline{\lambda}$ is a hyperasymptotic curve relative to $\underline{\mu}$ and the conditions mentioned above (in case 2) are satisfied, then the ratios $k_{(1)}/K_{(1)}$ and $K_n/K_{(1)}$ are the same at each point of the curve.

Case 3. The condition stated in case1 and 2 are not satisfied, i.e., the matrix A is singular, $\Omega_{ij}\xi^i_{(1)}\frac{ds^j}{ds}$ may or may not be zero.

We have $DK_n + k_{(1)}S\cos\theta = 0$, which in view of (2.7) yields the following

Theorem 4.2. If a K_λ curve is hyperasymptotic relative to $\underline{\mu}$ and the condition mentioned in case 1 and 2 are not satisfied, then

$$\left.\begin{aligned} k_{(1)} &= \frac{K_{(1)}\left(1-S^2\right)^{1/2}}{\left(1-S\sin^2\theta\right)^{1/2}}, \\ K_n &= \frac{-K_{(1)}S\cos\theta}{\left(1-S^2\sin^2\theta\right)^{1/2}} \end{aligned}\right\} \qquad ...(4.3)$$

Comparing (4.3) and (2.8), we have the following

Theorem 4.3. If two special curves Γ and $\bar{\Gamma}$ relative to $\underline{\lambda}$ are respectively the union and hyperasymptotic curves relative to another congruence $\underline{\mu}$ and the conditions stated in case 1 and 2 are not satisfied, then the modulus of the normal curvature in the direction of Γ is equal to the first curvature of $\bar{\Gamma}$ and the first curvature of Γ is equal to the modulus of the normal curvature of $\bar{\Gamma}$.

REFERENCE

[1] R.S. Mishra , A *Course in Tensor with Appllication to Riemannian Geometry*, Pothishala Private Limited (1965).

[2] R.S. Mishra , Hyperasymptotic curves on a Riemannian hyper-surface, *Math. Student*, **20**, (1952), 63-65.

[3] U.P. Singh, On sheeial union and hyherasymptatic curves is Riemannian spaces, Tensor(N.S.), 22 (1971), 15-18.

[4] C. Springer, Union curves of a hypersurface, *Can J. Math,* **2** (1950), 457-460.

[5] G. Tsagas , Special curves of a hypersurface of Riemannian space, *Tensor*, *N.S.*, **20**, (1969), 88-90.

Jñānābha, Vol. 38, 2008

MEROMORPHIC UNIVALINT FUNCTIONS WITH ALLTERNATING COEEFICIENTS

By

K.K. Dixit and Indu Bala Mishra

Department of Mathematics,

Janta College, Bakewar, Etawah-206124, Uttar Pradesh, India

Emaial : dr-indubalamishra@yahoo.com

(Received : March 8, 2008)

ABSTRACT

Let $T_M(A,B,z_0)$ denote the subclass of functions $f(z)=\frac{a}{z}+\sum_{n=1}^{\infty}(-1)^{n-1}a_n z^n$ $(a \geq 1, a_n \geq 0)$ regular and univalent in the disk $D=\{z:0<|z|<1\}$ with a simple pole at $z=0$ the conditions $\left|\frac{\frac{zf'(z)}{f(z)}+1}{A+Bz\frac{f'(z)}{f(z)}}\right| \leq 1, z \in D$ and $f'(z_0)=-\frac{1}{z_0^2}$ where $0<z_0<1$.

Sharp coefficients estimates, radius of meromorphic convexity, integral transform of functions for this class have been obtained. It is also seen that the class $T_M(A,B,z_0)$ is closed under convex linear combination and in the last, certain convolution properties of functions have been studied,

2000 Mathematics Subject clasification : Primary 32H04; Secondary 32E*xx*.

Keywords : Meromerphic Univalent function, meromorphic cenvexity.

1. Introduction. Let Σ be the class of functions of the form $f(z)=1/z+a_1z+a_2z^2+\ldots$ that are regular and univalent in punctured disk $D=\{z:0<|z|<1\}$ with a simple pole z=0. A function *f* in Σ is said to be starlike of order $\alpha(0 \leq \alpha < 1)$ denoted by $f \in \sum^*(\alpha)$ if $\operatorname{Re}\frac{zf'(z)}{f(z)}<-\alpha$ for $|z|<1$. The class $\sum^*(\alpha)$ has been extensively studied by Bajpai [1], Cluni [2], Goel and Sohi [4], Juneja and Reddy [5], Silverman [6] and others.

Let Σ_m be the subclass of Σ consisting of functions of the form

$$f(z)=\frac{1}{z}+a_1z-a_2z^2+a_3z^3+\ldots,a_n\geq 0$$

$$=\frac{1}{z}+\sum_{n=1}^{\infty}(-1)^{n-1}a_nz^n,\quad a_n\geq 0 \qquad \ldots(1.1)$$

Uralgaddi and Ganigi [7] have obtained certain results for meromorphic functions with alternating coefficients that are starlike of order α.

Let $\sum_M(A,B)$ denote the class of functions of the form (1.1) regular and univalent in the punctured disk D satisfying the condition

$$\left|\frac{\frac{zf'(z)}{f(z)}+1}{A+B\frac{zf'(z)}{f(z)}}\right|\leq 1, z\in D.$$

Dixit and Misra [3] have determined certain interesting results for the class $\sum_M(A,B)$, where A and B are fixed numbers $-1\leq A<B\leq 1, 0\leq B\leq 1$.

Let T_M denote the class of functions $f(z)=\frac{a}{z}+\sum_{n=1}^{\infty}(-1)^{n-1}a_nz^n(a_1\geq 1, a_n\geq 0)$ regular and univalent in the disk D. Let $T_M(A,B)$ denote the subclass of functions in T_M satisfying the condition.

$$\left|\frac{\frac{zf'(z)}{f(z)}+1}{A+B\frac{zf'(z)}{f(z)}}\right|\leq 1,\quad z\in D. \qquad \ldots(1.2)$$

Also $T_M(A,B,z_0)$ denote the subclass of functions in $T_M(A,B)$ satisfying $f'(z_0)=-1/z_0^2$ where $0<z_0<1$.

In this paper, we obtain sharp coefficient estimated, radius of meromorphic convexity, integral transform of functions in $T_M(A,B,z_0)$. It is also shown that the class $T_M(A,B,z_0)$ is closed under convex linear combination. In the last, convolution problem of functions have been studied.

We begin by recalling the following lemma due to Dixit and Misra [3].

Lemma: A function f in $\Sigma M'$ is in $\Sigma M\ (A,B)$ if and only if

$$\sum_{n=1}^{\infty}\{n(1+B)+1+A\}a_n \le B-A. \qquad ...(1.3)$$

2. Coefficient Estimates

Theorem 2.1. Let $f(z)=\frac{a}{2}+\sum_{n=1}^{\infty}(-1)^{n-1}a_n z^n (a_1 \ge 1, a_n \ge 0)$. If f is regular in D and satisfies $f'(z_0)=-1/z_0^2$, then $f \in T_M(A,B,z_0)$ if and only if

$$\sum_{n=1}^{\infty}\left[\{n(1+B)+1+A\}-n(B-A)z_0^{n+1}(-1)^{n-1}\right]a_n \le B-A. \qquad ...(2.1)$$

The result is sharp.

Proof. We know from lemma (1.3) that a function $g(z)=\frac{1}{z}+\sum_{n=1}^{\infty}(-1)^{n-1}b_n z^n, (b_n \ge 0)$ regular in D, satisfies

$$\left|\frac{\frac{zg'(z)}{g(z)}+1}{Bz\frac{g'(z)}{g(z)}+A}\right| \le 1, \qquad z \in D.$$

if and only if $\sum_{n=1}^{\infty}[\{n(1+B)+1+A\}b_n] \le B-A.$ ...(2.2)

Applying the result to the function $g(z)=f(z)/a$, we find that f satisfies (1.3) if and only if $\sum_{n=1}^{\infty}[\{n(1+B)+1+A\}a_n] \le a(B-A).$ ...(2.3)

Since $f'(z)=-1/z_0^2$, we also have from the representation of $f(z)$ that

$$a=1+\sum_{n=1}^{\infty}(-1)^{n-1}na_n z_0^{n+1}$$

Putting the value of a in inequality (2.3), we have the required result.
For attaining the equality (2.1), we choose the function

$$f(z)=\frac{\{n(1+B)+1+A\}\frac{1}{z}+(-1)^{n-1}(B-A)z^{n}}{n(1+B)+1+A-(-1)^{n+1}n(B-A)z_0^{n+1}} \quad \text{...(2.4)}$$

From (2.4), we have

$$a_n=\frac{B-A}{\{n(1+B)+1+A\}-(-1)^{n-1}n(B-A)z_0^{n+1}}$$

or $\quad \left[\{n(1+B)+1+A\}-(-1)^{n-1}n(B-A)z_0^{n+1}=B-A,\right]$

and $\quad a=1+\sum_{n=1}^{\infty}(-1)^{n-1}na_n z_0^{n+1}$

$$=1+\frac{(-1)^{n-1}n(B-A)z_0^{n+1}}{\{n(1+B)+1+A\}-(-1)^{n-1}n(B-A)z_0^{n+1}}$$

$$=\frac{\{n(1+B)+1+A\}}{\{n(1+B)+1+A\}-(-1)^{n-1}n(B-A)z_0^{n+1}}>1.$$

3. Radius of Meromorphically Convexity

Theorem 3.1. If $f\in T_M(A,B,z_0)$, then f is meromorphically convex of order $\delta(0\le\delta<1)$ in the disc $|z|<R$, where

$$R=\inf_{n>1}\left[\frac{(1-\delta)\{n(1+B)+1+A\}}{n(n+2-\delta)(B-A)}\right]^{\frac{1}{n+1}} \quad \text{...(3.1)}$$

This result is sharp for each n for functions of the form (2.4).

Proof. In order to determine the required result, is suffices to show that $\left|2+\frac{zf''(z)}{f'(z)}\right|\le 1-\delta(|z|<R)$ for a function $f(z)$ belonging to the class $T_M(A,B,z_0)$, where R is defined by (3.1). The details involved are fairly straight forward and may be omitted.

4. Integral Transform.

Theorem 4.1. If $f\in T_M(A,B,z_0)$, then the integral transform

$F(z)=c\int_0^1 u^c f(uz)du$ for $0<c<\infty$, is in $T_M(A',B',z_0)$, where

$$\frac{1+B'}{1-A'} \leq \frac{(A+B+2)(c+2)+(B-A)c}{Zc(B-A)} - \frac{z_0^2}{c}.$$

The result is sharp for the function

$$f(z) = \frac{(A+B+2)1/z+(B-A)z}{(A+B+2)-(B-A)z_0^2}.$$

Proof. Let $f(z) = \dfrac{a}{z} + \sum_{n=1}^{\infty}(-1)^{n-1}a_n z^n \in T_M(A,B,z_0)$,

then $\quad F(z) = c\int_0^1 \left[u^{c-1}\dfrac{a}{z} + \sum_{n=1}^{\infty}(-1)^{n-1}a_n u^{n+c}z^n\right]du$

$$= \frac{a}{z} + \sum_{n=1}^{\infty}(-1)^n \frac{a_n z^n}{n+c+1}.$$

It is sufficient to show that

$$\sum_{n=1}^{\infty} \frac{\{n(1+B')+1+A'\}-(-1)^n n(B'-A')z_0^{n+1}}{(B'-A')(n+c+1)} a_n \leq 1. \qquad \text{...(4.1)}$$

Since $f \in T_M(A,B,z_0)$ implies that

$$\sum_{n=1}^{\infty} \frac{\{n(1+B)+1+A\}-(-1)^{n-1} n(B-A)z_0^{n+1}}{(B-A)} a_n \leq 1.$$

From Theorem 2.1, (4.1) will be satisfied if

$$\frac{1+B'}{B'-A'} \leq \frac{\{n(1+B)+1+A\}(n+c+1)+(B-A)c}{(n+1)(B-A)c} - (-1)^{n-1}\frac{n}{c}z_0^{n+1} \qquad \text{...(4.2)}$$

The right hand side of (4.2) is an increasing function of n, therefore putting $n=1$ in (4.2) we have

$$\frac{1+B'}{B'-A'} \leq \frac{(A-B+2)(c+2)+(B-A)c}{2c(B-A)} - \frac{z_0^2}{c}.$$

Hence the theorem.

5. Closure Theorems

Theorem 5.1. Let γ be a real number such that $\gamma > 1$. If $f \in T_M(A,B,z_0)$, then the function F defined by

$$F(z)=\frac{(\gamma-1)}{z^{\gamma}}\int_0^z t^{\gamma-1}f(t)dt\ .$$

Also belong to $T_M(A,B,z_0)$.

Proof . The proof of Theorem 5.1 is similar to that of Dixit and Misra [3].

Since the class $T_M(A,B,z_0)$ is convex, it does have some 'extreme points' given by Theorem 5.2.

Theorem 5.2. Let $f(z)=\dfrac{1}{z}$ and $f_n(z)=\dfrac{\{n(1+B)+1+A\}\frac{1}{z}+(B-A)(-1)^{n-1}z^n}{\{n(1+B)+1+A\}-n(B-A)(-1)^{n-1}z_0^{n+1}}$.

Then $h\in T_M(A,B,z_0)$ if and only if it can be expressed in the form

$$h(z)=\lambda f(z)+\sum_{n=1}^{\infty}\lambda_n f_n(z)$$

where $\lambda\geq 0$ and $\lambda+\sum_{n=1}^{\infty}\lambda_n=1$.

Proof : Let us suppose that

$$h(z)=\lambda f(z)+\sum_{n=1}^{\infty}\lambda_n f_n(z)$$

$$=\frac{a}{z}+\sum_{n=1}^{\infty}(-1)^{n-1}a_n z^n$$

where $a=\lambda+\sum_{n=1}^{\infty}\dfrac{\{n(1+B)+1+A\}\lambda_n}{\{n(1+B)+1+A\}-(-1)^{n-1}n(B-A)z_0^{n+1}}$.

and $a_n=\dfrac{(B-A)\lambda_n}{\{n(1+B)+1+A\}-(-1)^{n-1}n(B-A)z_0^{n+1}}$

Then, it is easy to see that $h'(z_0)=-1/z_0^2$ and the condition (2.1) is satisfied. Hence $h\in T_M(A,B,z_0)$. Conversely, Let $h\in T_M(A,B,z_0)$ and

$$h(z)=\frac{a}{z}+\sum_{n=1}^{\infty}(-1)^{n-1}a_n z^n$$

Then, from (2.1), it follows that

$$a_n \leq \frac{(B-A)}{\{n(1+B)+1+A\}-(-1)^{n-1}n(B-A)z_0^{n+1}} \qquad (n=1,2,3...).$$

Setting

$$\lambda_n = \left[\frac{\{n(1+B)+1+A\}-(-1)^{n-1}n(B-A)z_0^{n-1}}{B-A}\right]a_n$$

and

$$\lambda = 1-\sum_{n=1}^{\infty}\lambda_n,$$

we have

$$h(z) = \lambda f(z)+\sum_{n=1}^{\infty}\lambda_n f_n(z).$$

This completes the proof of the theorem.

The following inclusion property is an easy consequence of the Theorem 2.1.

Theorem 5.3. Let $f_j(z)=\frac{a_j}{z}+\sum_{n=1}^{\infty}(-1)^{n-1}a_{nj}z^n, j=1,2,...,m.$ If $f_j \in T_M(A,B,z_0)$ for each $j=1,2,...,m$, then the function

$$h(z)=\frac{b}{z}+\sum_{n=1}^{\infty}(-1)^{n-1}b_n z^n \text{ also belongs to } T_M(A,B,z_0), \text{ where}$$

$$b=\sum_{j=1}^{m}\lambda_j a_j, \quad b_n=\sum_{j=1}^{\infty}\lambda_j a_{nj}, \qquad (n=1,2,...,m)$$

$$\lambda_j \geq 0 \text{ and } \sum_{j=1}^{\infty}\lambda_j = 1.$$

Theorem 5.4. If $f(z)=\frac{a}{z}+\sum_{n=1}^{\infty}(-1)^{n-1}a_n z^n \in T_M(A,B,z_0)$ and $g(z)=\frac{b}{Z}+\sum_{n=1}^{\infty}(-1)^{n-1}b_n z^n$ with $b_n \leq 1$ for n=1,2,3..., then $f * g \in T_M(A,B,z_0)$.

Proof. For convolution of functions *f* and *g*; we can write

$$\sum_{n=1}^{\infty}\left[\{n(1+B)+1+A\}-(-1)^{n-1}n(B-A)z_0^{n+1}\right]a_n b_n \le \sum_{n=1}^{\infty}\left[\{n(1+B)+1+A\}-(-1)^{n-1}n(B-A)z_0^{n+1}\right]a_n$$

(Since $b_n \le 1$)

$\le$ B-A, by (2.1)

Hence, by theorem (2.1), $f * g \in T_M(A,B,z_0)$.

Note. It will be of interest to find other convolution results analogous to those of Juneja Reddy [5].

REFERENCES

[1] S. K., A note on a class of meromorphic univalent functions, *Rev. Roumanie Maths Puse Appl.*, 22 (1977). 295-297.

[2] J. Clunie, On meromorphic Schlicht functions, *J. London Math. Soc.*, **34** (1959) 215-216.

[3] K.K. Dixit and Y.K. Misra, On a class of meromorphic starlike functions with alternating coefficients, *Acta Cientia Indica*, **XXVII M, No(1)** (2001) 065.

[4] R.M. Goel and N.S. Sohi, On class of Meromorphic function, Glasnik Mathematicki, **17** (1981), 19-28.

[5] O.P. Juneja and T.R. Reddy, Meromorphic starlike univalent functions with positive coefficients, *Annales Universitatis Mariae Curie- Sklodawska Lubin-Polonia*, **39** (9) **Section** A (1985) 55-75.

[6] H. Silverman , Univalent functions with negative coefficients, *Proc. Amer. Math Soc*. **51** (1975) 109-116.

[7] B. A. Uralegaddi and M.D. Ganigi, Meromorphic starlike functions with alternating coefficients, *Rendiconti di Mathematica, Seril VII*, Vol. **II**, Roma (1991), 441-446.

Jñānābha, Vol. 38, 2008

A MATHEMATICAL MODEL FOR MIGRATION OF CAPILLARY SPROUTS WITH DIFFUSION OF CHEMOTTRACTANT CONCENTRATION DURING TUMOR ANGIOGENESIS

By

Madhu Jain, G. C. Sharma

Department of Mathematics, Institute of Basic Science,

Dr B. R. Ambedkar University, Khandari, Agra-282002, Uttar Pradesh, India

Email : madhujain@sancharnet.in, gokulchandra@bsnl.in

and

Atar Singh

Department of Mathematics, Bipin Bihari College, Jhansi-284128, U.P., India

(Received : March 12,2008)

ABSTRACT

In this paper, we develop a simplified mathematical model for tumor angiogenesis, which is the process of sprout growth, forming new blood vessels from already existing micro vessels. The diffusion of tumor angiogenesis factor in extra cellular matrix is explored and the effects of *TAF*, chemottractant concentration, on tip density and vessel density have also been discussed. The analytical solution has been obtained. The numerical experiments have been performed to justify the same and to facilitate sensitivity analysis.

2000 Mathematics Subject Classification : Primary 92C10 Secondary 92C37.

Keywords: Angiogenesis, Tumor growth, Diffusion, *TAF*, Neovascularization.

1. Introduction. The studies on cancer are being made due to its prevalence all over the world. Very recently a known scientist Sasi Sekharan from *MIT*, *USA* has tested the molecular size bomb on the skin or lung cancer of the mice. Basically, solid tumors are the collection of tumor cells, which are the core factors in cancer research. Other factors like neovascularization and the interstitium are also accountable for solid tumor growth. The aim of all the studies in this area is to find a drug, which can penetrate deep into tumors, cut of the blood supply and detonate a lethal dose of anticancer toxins without harming healthy cells. Solid tumors are of two types, avascular and vascular. Initially solid tumors are small mass cells within the tissue. But this mass multiplies rapidly due to the process of multation. At initial stage, avascular tumor growth leads to non-metastatic tumor, which may remain dormant for long time. At avascular stage, tumor growth may not be possible beyond 1-2 millimeters.

For tumor growth, vascularization to tumor is one of the important factors. This process occurs both in non-malignant conditions. Angiogenesis is the process

of sprouting new blood vessels from already exsting micro vessels. To make the tumor vascularized, solid tumor secrets some chemical compounds, which stimulate endothelial cells (EC) of the tissue. Some enzymes secreted by EC, break down basement of membrane, allowing endothelial cells to proliferate across the disrupted membrane into extra cellular matrix. On the basis of this process, new capillary sprouts form and migrate towards a vascular solid tumor. Finally, a network of capillaries is formed and penetrates into the tumor. Then a solid tumor gets vascularization. Many researchers developed mathematical models related to tumor growth in different frame works.

Muthu et al. (1982) studied tumor induced-neovascularization in mouse eye. Chaplain and Sleemen (1990) developed a mathematical model for prodution and secretion of tumor angiogenesis factor in tumors. Adam and Maggelakis (1990) explored diffusion regulated growth characteristics of spherical pre-vascular carcinoma. Lauffenburger and Stokes (1991) analysed the roles of microvessel endothelial cell random motility and chemotaxis in angiogenesis. Chaplain and Stuart (1993) provided a mechanism for chemotrctic response of endothelial cells to tumor angiogenesis. Byrne and Chaplain (1999) developed a model of vascular solid tumor growth. Cui and Friedman (2001) developed a mathematical model of the growth of necrotic tumors. Friedman and Reitch (2001) worked on the existence of spatially patterned dormant malignancies in a model for the growth of non-necrotic vascular tumor. Levine et al. (2001) discussed onset of capillary formation initiating angiogenesis. Petted et al. (2001) studied the migration of cells in multicells tumor angiogenesis. Petted et al. (2001) studied the migration of cells in multicells tumor spheroids. McDougall et al. (2002) examined mathematically the flow through networks: in order to study tumor-induced angiogenesis and chemotherapy strategies. Bazaliy and Friedman (2003) prepared a model of tumor growth. Cui and Friedman (2003) studied a hyperbolic free boundary problems modeling tumor growth. Owen et al. (2004) did work on mathematical modeling of the use of macrophages as vehicles for drug delivery to hypoxic tumor sites. They investigated the role of chemotaxis and chemokine production and the efficacy of macrophages as vehicles for drug delivery. Plank et al. (2004) formulated a mathematical model of tumor angiogenesis, regulated by vascular endothelial growth factor and the angiopoietins. this work attempted to provide a mathematical description of the role of the angiopoietins in angiogenesis. Tarabolette and Giavazzi (2004) suggested modeling approaches for angiogenesis and designed for anti-angiogenic compounds.

The paper mathematical modeling is capable of providing deep insight into the ways by which one can think of producing some medicine to destroy the blood

vessels feeding the tumor. In this paper, we have modelled sprout growth during tumor angiogenesis and diffusion of TAF in extra cellular matrix. Our focus in the present investigation is on capillary sprout growth, diffusion of *TAF* and role of angiogenesis in tumor growth. The simplified model is solved analytically. Byrne and Chaplain (1996) have not considered the diffusion of tumor angiogenesis factor in their studies. Our effort in this paper is to extend their studies for more complicated cases by incorporating this factor. We examine the diffusion of chemottractant concentration , the role of angiogenesis in tumor growth. the numerical experiments have also been performed to visualize the effects of various parameters on tip and vessel densities. The present investigation is organized in the following way. In section 2, we describe mathematical model by stating requisite assumptions and notations. The analysis part of the paper has been discussed in section 3. The numerical illustrations are presented to validate analytical results in section 4. The section 5 provides descussion part and concluding remarks.

2. The Model. It is well known that vessels within a sprout have a velocity forming the continuity of the tip-vessels structure. Tumor angiogenesis factor (*TAF*) concentration is governed by reaction-diffusion equation based on the experimental·observation and earlier studies. We consider the finite and uni-directional model for sprout growth and diffusion to tumor. This model is formulated in terms of tip density, vessel density and *TAF* concentration. The tumor is at $x=0$ and limbus is at $x=1$, $(0<=x<=1)$. Following Byrne and Chaplain (1996) model , that vascular front with tips stimulated by *TAF* and sprouting from capillary vessels. The proliferation of tip occurs at the vascular edge (13) and is triggered when exceeded threshold concentration c'. The vessels within the sprouts migrate to tumor and decay linearly. For *TAF* concentration, reaction-diffusion equation with linear-decay has been used. We use the following notations for mathematical formulation of the problem.

The set of partial differential equations governing the model (Byrne and Chaplain, 1996) is given by

$$\frac{\partial n}{\partial t} = \mu \frac{\partial^2 n}{\partial x^2} - \chi \frac{\partial}{\partial x}\left(n \frac{\partial c}{\partial x}\right) + \alpha_0 \rho c + \alpha_1 H(c - c')nc - \beta n \rho \qquad ...(1)$$

$$\frac{\partial \rho}{\partial t} = \mu \frac{\partial n}{\partial x} - \chi n \frac{\partial c}{\partial x} - \gamma \rho \qquad ...(2)$$

$$\frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2} - \lambda c \, , \qquad ...(3)$$

where $n(x,t)$, $\rho(x,t)$, $c(x,t)$ are the density, vessel density and *TAF* concentration,

respectively. D and H are the diffusion coefficient of TAF and Hevinside unit function. $\lambda, \gamma, c, \beta, \mu, \chi, \alpha_0, \alpha_1$ are rate of TAF decay, rate of vessel density decay, threshold TAF concentration, rate of tip to branch anastomoses, random tip motility, chemo taxis coefficient, first tip proliferation rate, second proliferation rate respectively.

At the tumor, c is at constant value 1 and $n=0$, if tips penetrate the tumor, then our model is not applicable. At the limbus, $c=0$, while n and ρ decay exponentially to zero. If tip and capillary once formed then the capillary buds' production at limbus is stopped.

Initial and boundary conditions are

$$n(0,t)=0, \quad n(1,t)=n_c e^{-kt},\ n(x,0)=0, \text{ for } 0 \le x < 1, \quad n(1,0)=n_c \qquad \ldots(4)$$

$$\rho(1,t)=\rho_{\min}+(1-\rho_{\min})e^{-kt}, \quad \rho(x,0)=0, \text{ for } 0 \le x < 1, \quad \rho(1,0)=1 \qquad \ldots(5)$$

$$c(0,t)=1, c(1,t)=0 \quad c(x,0)=c_0 \quad \text{for } 0 \le x < 1, \qquad \ldots(6)$$

where k is decay rate of tip and vessel density.

3. The Analysis. The TAF concentration for unsteady profile with the help of the equations (3) and (6) can be obtained as:

$$c = 2c_0 e^{-\lambda t} \sum_{r=1}^{\infty} \frac{(-1)^n}{\beta_n} e^{-D\beta_n^2 t} \cos \beta_n x \qquad \ldots(7)$$

where $\beta_n = \pi\left(\dfrac{2n-1}{2}\right)$ and $n = 1,2,3,\ldots,\infty$

Chaplain and Stuart [9], estimated motility coefficient for cells chemicals $D_{cell}/D_{chemical} = \mu \approx 10^{-3} \approx 0$.

Then vessel density within the sprout is given by $\dfrac{\partial \rho}{\partial t} = -\chi n \dfrac{\partial c}{\partial x} - \gamma \rho \qquad \ldots(8)$

Solving the equation (8) with the help of the equation (5), we have

$$\rho = 2c_0 \chi e^{-\gamma t} \sum_{r=1}^{\infty} (-1)^r \sin \beta_n x \int e^{(\gamma - \lambda - D\beta_n^2 t)} n(x,t)\,dt \qquad \ldots(9)$$

The non-linear wave equation for tip density can be reduced to

$$\frac{\partial n}{\partial t} - \chi \frac{\partial n}{\partial x}\frac{\partial c}{\partial x} - \chi n \frac{\partial^2 c}{\partial x^2} + \alpha_0 c \rho + \alpha_1 H(c-c')nc - \beta n \rho \qquad \ldots(10)$$

With the help of equation (9), the equation (10) in terms of $n(x,t)$ and $c(x,t)$ can be written as

$$\frac{\partial n}{\partial t}=-\chi\frac{\partial n}{\partial x}\frac{\partial c}{\partial x}=\left(\alpha_1 H(c-c')-\chi\beta_n^2\right)nc-(\alpha_0 c-\beta n)$$

$$\times(-1)^r 2C_0\chi e^{-\gamma t}\sin\beta_n x\int e^{(\gamma-\lambda-D\beta_n^2)t}n(x,t)dt \qquad ...(11)$$

We simplify the model with the concept that the vessels form tips for a very short time and in view of this $\frac{\partial n}{\partial t}=0$, so that $\rho(x,t)=-\frac{\chi}{\gamma}\frac{dc}{dx}n.$

Now we have

$$\frac{\partial n}{\partial t}-\chi\frac{\partial n}{\partial x}\frac{\partial c}{\partial x}=\left(\alpha_1 H(c-c')+\chi\beta_n^2-\chi\frac{\alpha_0}{\gamma}\frac{dc}{dx}\right)cn+\frac{\chi\beta}{\gamma}\frac{dc}{dx}n^2 \qquad ...(12)$$

With the condition

$n(0,t)=0 \qquad n(x,0)=H(x-x^*),\ n(1,t)=1$ for some $x^*\in(0,1)$

we further simplify the model in respect of *TAF* concentration and tip creation rates. For more simple approximation, if we consider,

$\lambda=0, \qquad \alpha_0=0, \qquad H(c-c')=1.$

The reduced equation is given as follows:

$$\frac{\partial n}{\partial t}+\chi\frac{\partial c}{\partial x}\frac{\partial n}{\partial x}=\alpha_1 nc+\chi\beta_n^2 nc+\frac{\chi\beta}{\gamma}\frac{dc}{dx}n^2 \qquad ...(13)$$

We use perturbation method for approximation of $n(x,t)$, which provides

$$n(x,t)=n_0(x)+\in n_1(x)e^{i\mu t}+\in^2 n_2(x)e^{2i\mu t}+... \qquad ...(14)$$

where $\in$ is a small quantity.

Now

$$\frac{\partial n}{\partial t}=\in n_1 i\mu e^{i\mu t}, \qquad \frac{\partial n}{\partial x}=\frac{\partial n_0}{\partial x}+\in e^{i\mu t}\frac{\partial n_1}{\partial x} \qquad ...(15)$$

From equations (14) and (15), we get

$$\in n_1 i\mu e^{i\mu t}+2\chi C_0(-1)^r e^{-\alpha t}\sin\beta_n x\left(\frac{\partial n_0}{\partial x}+\in e^{i\mu t}\frac{\partial n_1}{\partial x}\right)$$

$$=-2c_0(-1)^r\alpha_1 e^{-\alpha t}\cos\beta_n x\left(n_0+\in n_1 e^{i\mu t}\right)/\beta_n$$

$$-2\chi\beta_n C_0(-1)^r e^{-\alpha t}\cos\beta_n x\left(n_0+\in n_1 e^{i\mu t}\right)$$

$$+\frac{2\beta\chi C_0(-1)^r e^{-\alpha t}\sin\beta_n x}{\gamma}\left(n_0^2+\in^2 n_1^2 e^{2i\mu t}+2n_0 n_1 e^{i\mu t}\right) \qquad ...(16)$$

where $\alpha=D\beta_n^2$.

Comparing terms free of $\in$ on both sids, we have

$$\frac{\partial n_0}{\partial x} = -\frac{\alpha_1}{\chi\beta_n}\cot\beta_n x.n_0 + \frac{\beta}{\gamma}n_0^2 - \beta_n\cot\beta_n x.n_0$$

or $$\frac{\partial n_0}{\partial x} = -F\cot\beta_n x.n_0 + F' n_0^2$$

where $F = -\left(\frac{\alpha_1}{\chi\beta_n} + \beta_n\right)$ and $F' = \frac{\beta}{\gamma}$.

On solving, equation (17) with the help of (4), we get

$$n_0 = \frac{(\sin\beta_n x)^{-G}}{-F'\int(\sin\beta_n x)^{-G}dx}$$

where $G = \frac{\alpha_1 + \chi\beta_n^2}{\chi\beta_n^2}$.

Now comparing the coefficients of $\in$ on both sides. we have

$$n_1 i\mu + 2\chi C_0(-1)^r e^{-\alpha t}\sin\beta_n x.\frac{\partial n_1}{\partial x} = 2C_0(-1)^r e^{-\alpha t}\left(\chi\beta_n^2 + \alpha_1\right)\frac{\cos\beta_n x}{\beta_n}$$

$$+2C_0(-1)^r e^{\alpha t}\beta\chi\sin\beta_n x.\frac{n_0 n_1}{\gamma}$$

or $$(p\chi\sin\beta_n x)\frac{dn_1}{dx} = \left[\left(\frac{p\beta\chi}{\gamma}n_0\sin\beta_n x - \left(\frac{\chi\beta_n^2 + \alpha_1}{\beta_n}\right)p\cos\beta_n x\right) - i\mu\right].n_1 \quad ...(18)$$

On solving the differential equation (18), we get

$$\log n_1 = -\int\frac{(\sin\beta_n x)^{-G}}{\int(\sin\beta_n x)^{-G}dx}dx - \log(\sin\beta_n x)^G - \frac{P}{\beta_n}\log(\tan\beta_n x/2) + C \quad ...(19)$$

With the help of the (4), then we have

$C=0$,

$$\log n_1 = -\int\frac{(\sin\beta_n x)^{-G}}{\int(\sin\beta_n x)^{-G}dx}dx - \log(\sin\beta_n x)^G - \frac{P}{\beta_n}\log(\tan\beta_n x/2) \quad ...(20)$$

$$n_1 = e^{-\int\frac{(\sin\beta_n x)^{-G}}{\int(\sin\beta_n x)^{-G}dx}dx - \log(\sin\beta_n x)^G - \frac{P}{\beta_n}\log(\tan\beta_n x/2)} = e^B \quad ...(21)$$

where $B = -\int\frac{(\sin\beta_n x)^{-G}}{\int(\sin\beta_n x)^{-G}dx}dx - \log(\sin\beta_n x)^G - \frac{P}{\beta_n}\log(\tan\beta_n x/2)$.

Now $\quad p = 2C_0(-1)^n e^{-\alpha t}, \qquad P = \dfrac{i\mu}{p\chi}$

$$n(x,t) = n_0 + \in e^{i\mu t} e^B + \ldots \qquad \ldots(22)$$

Substituting the values of n_0 and n_1 in (22), we have

$$n(x,t) = \frac{(\sin\beta_n x)^{-G}}{-F' \int (\sin\beta_n x)^{-G} dx} + \in e^{i\mu t}$$

$$\times \exp\left(-\int \frac{(\sin\beta_n x)^{-G}}{\int (\sin\beta_n x)^{-G} dx} dx - \log(\sin\beta_n x)^G - \frac{P}{\beta_n} \log(\tan\beta_n x/2) \right) + \ldots$$

4. Numerical Results. The effect of various parameters on *TAF*, vessel density and tip density has been examined by considering a numerical illustration. We fix default parameters as $c_0 = .01, \lambda = .8, r = 1, \chi = .4, \alpha_1 = 1, \beta = 100,\ \ \mu = .001,$ $\gamma = .000002,$ to obtain numerical results, which are depicted in figures 1-4.

Figs. 1(a-c) depict that chemottractant concentration (c) varying x for different rate of diffusion coefficients (D) for t=0.8, 1,2, 1.6, respectively. It is noticed that the concentration decreases by increasing x and it is highest where tumor is situted (i.e. x=0) and is almost zero at limbus (i.e. x=1). The chemottractant concentration decreases as the rate of diffusion increases. Initially the concentration decreases slightly upto x=0.2 but beyond this, it decreases sharply. On comparing the figs. 1(a-c) we see that the concentration decreases as time increases; difference of concentration for different values of D increases as time increases but diminishes towards limbus.

Figs. 2(a-c) demonstrate the chemottactant concentration profile for different values of x. We see that concentration decreases as time increases and distance of limbus from tumor (x) increases. It is also clear from the figures that the concentration, at D=0.3 for t=2.8 and for different values of x, is almost zero; however for lower value fo D depicted in figures 2 (a)& 2 (b) for D=0.01 and D=0.1, respectively, the value of c becomes almost zero much later. Thus the diffusion coeficient has significant effect in the growth of the tumor.

It figures 3 (a-c); the effects of different parameters on vessel density (ρ) are depicted. Fig.3(a) illustrates vessel density (ρ) vs. x (distance of limbus from tumor) at different time. We observe that at limbus (x=1) the vessel density is highest and at tumor site (x=0) the vessel density is zero; the vessel density increases with time but the effect is more prevalent for higher values of x. In fig 3(b), we see that the vessel density increases initially sharply with x upto x=0.6, but beyond that, the density increases gradually with x. It also increases as χ,

increases. In fig. 3(c), which shows the vessel density profile for different values of χ, initially the vessel density increases slightly with time but later on it increases sharply. Upto $t=0.4$, the vessel formation for all values of χ is very slow and beyond that, the vessel formation increases sharply as χ increases.

Figs. 4(a-c) demonstrate tip density (n) $vs.$ for different values of t and χ. From fig. 4(a) we see that tip density increases steeply with x upto $x=0.7$ but beyond, the tip formation is static upto $x=0.8$ and onwards tip formation decreases. The tip density does not change with time; the path of tip formation is of wave type. Fig. 4(b) exhibits that tip density (n) increases as χ and x increases upto $x=0.7$, beyond this, its increasing trend slows down and converges at limbus (at $x=1$) for different values of χ. From the fig. 4(c), it is clear that tip density is free of time and increases as χ increases.

From numerical experiment performed concluding we infer that-

- The chemottractant concentration decreases as x,D and t increase; higher falling trends for lower rate of diffusion is noticed. For very long time, the concentration is static. Thus higher rate of diffusion and increasing trend in time decrease the chemottractant concentration, which is quite natural in real life situation.
- The effect of diffusion coefficient is more prevalent on chemottractant concentration.
- The vessel density is highest at limbus and decays to zero from limbus to the tumor. Also it decays as time increases and x decreases.
- Initially, the tip density at limbus is highest but it sharply decreases towards tumor and is almost zero at the tumor. It is also static with time.
- The effect of χ on the tip density and vessel density is quite remarkable. The path of tip density is in the form of wave propagation.

5. Conclusion. We have formulated a simplified mathematical model of capillary sprout growth with diffusion for *TAF* concentration for cancer cells to examine its ability to induce vascularization of the tumor in order to receive oxygen and nutrients. We make some simplifications to find out the effects of tip density and vessel density on tumor angiogenesis. The tip density and vessel density depends on *TAF* concentration. The diffusion is considered and its effects are discussed on *TAF*. Also the effects of various parameters on tip density and vessel density have been explained. It is very clear from the result that the *TAF* concentration depends on the rate of diffusion.

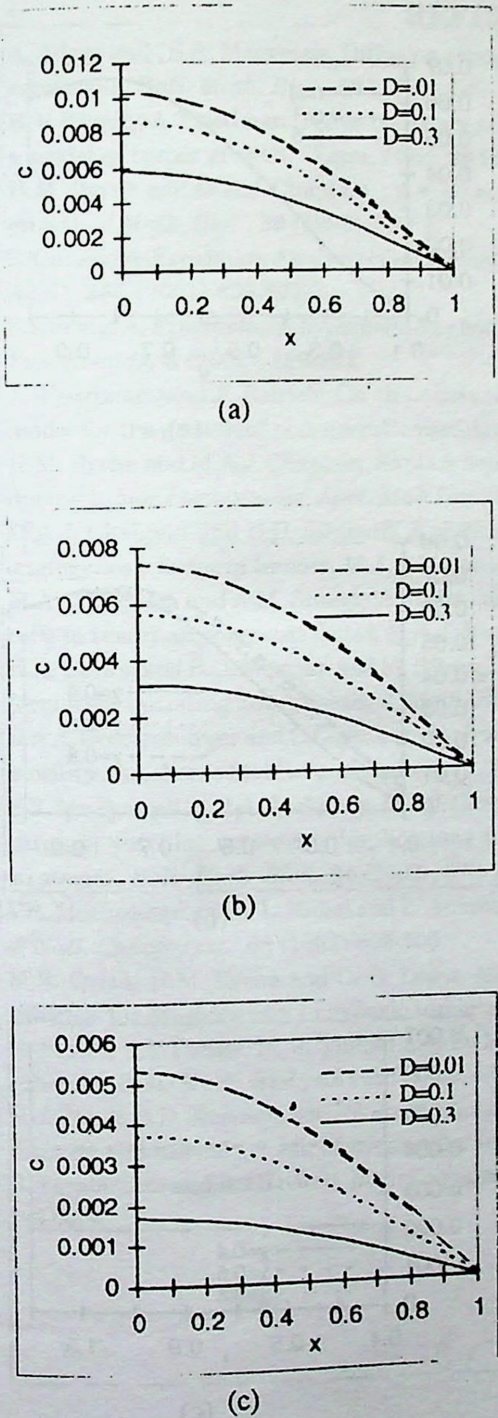


Fig. 1: Chamottractant concentration vs. x for different values of D and (a) t=0 .8, (b) t = 1.2, (c) t= 1.6

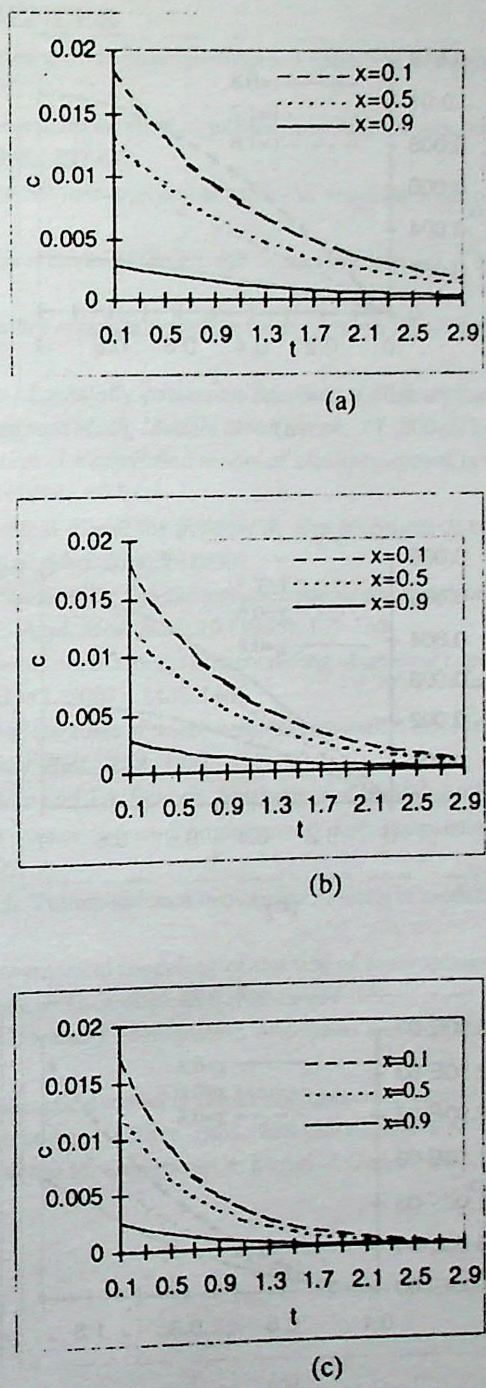


Fig. 2: Chamottractant concentration vs. t for different values of x and (a) D = 0.01, (b) D = 0.1, (c) D= 0.3

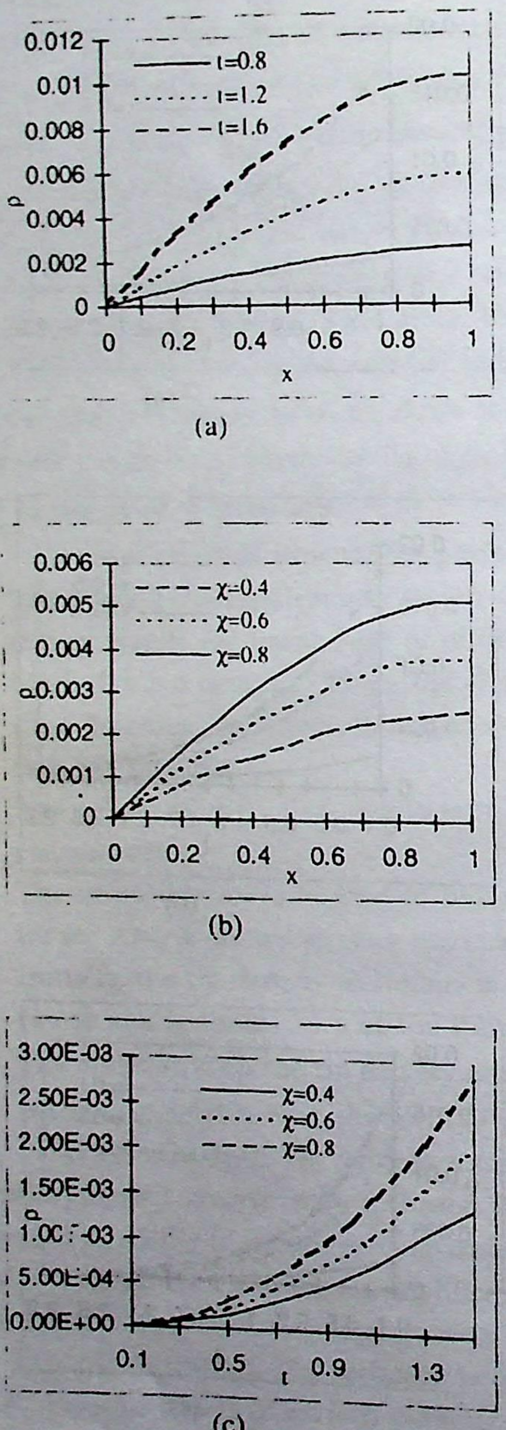


Fig. 3: Vessel density vs. x for different values of (a) t and (b) χ, at t=0.8 (c) ρ vs. t for different values of χ, at fixed value of x=0.1

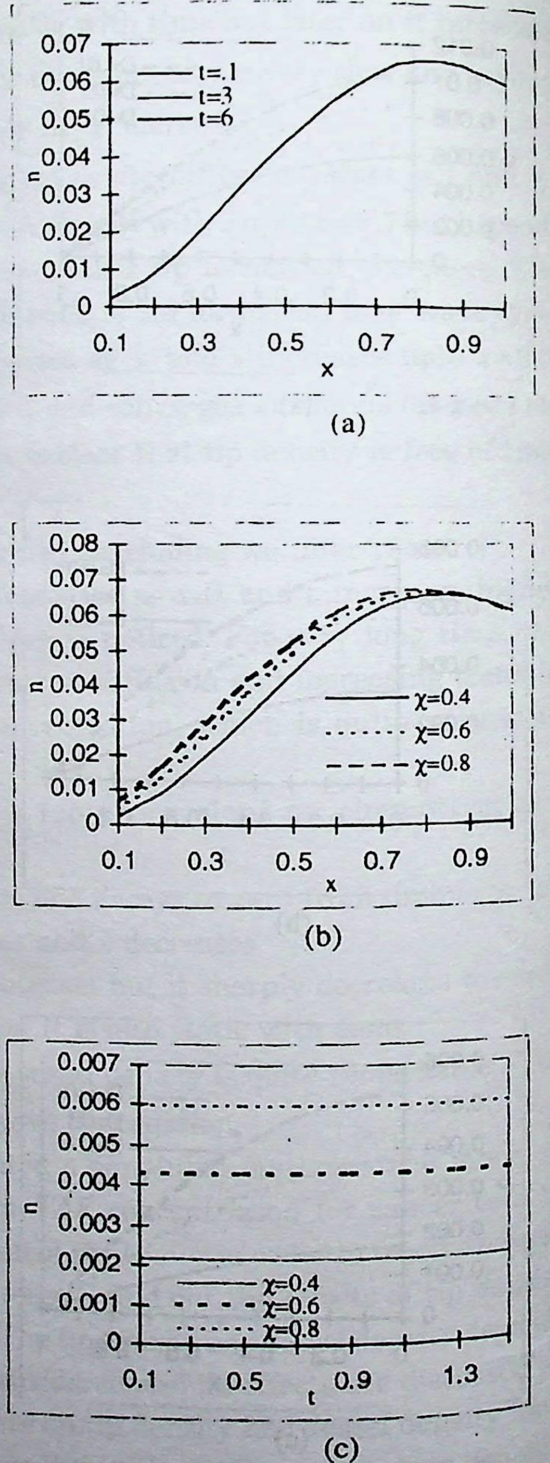


Fig. 4: Tip density vs. x for different values of (a) t and (b) χ, (c) ρ vs. t for different values of χ

REFERENCES

[1] A. Adam and , S.A. Maggelais, Diffusion regulated growth characteristics of spherical pre-vascular carcinoma, *Bull. Math. Biol.*, **52** (1990), 549-582.

[2] B. V. Bazaliy, A. Friedman , A free boundary problem for an elliptic- parabolic system: Application to a model of tumor growth, *Comm. PDE,* **28** (2003), 627-675.

[3] H.M. Byrne and M.A.J. Chaplain, : A weakly non-linear analysis of model of vascular solid tumar growth, *J.Math. Biol.*, **39** (1999). 59-89.

[4] S. Cui and A. Friedman, Analysis of matheamatical model of the growth of necrotic tumors, *J. Math. Anal.,* **255** (2001) 636-677.

[5] S. Cui and A. Friedman,: A hyperbolic free boundary problem modeling tumor growth, *Int-Faces and Free Bounds,* **5** (2003), 159-182.

[6] A. Friedman, and F. Reitich: On the existence of spatially patterned dormant malignancies in a model for the growth of non-necrotic vascular tumor, *Math. Models Mach. Appl.,* **77** (2001), 1-25.

[7] H.M. Byrne and M.A.J. Chaplain, Explicit solution of a simplified model of capillary sprout growth during tumor angiogenesis, *Appl. Mah.Lett.,* **9** (1996), 69-74.

[8] M.A.J. Chaplain and B.D. Sleeman, A matematical model for producton and secretion of tumor angiogenesis factor in tumors, *M.A.J. Maths. Appl. Med. Biol.,* **7** (1990).

[9] M.A.J.Chaplain and A.M. Stuart, A model mechanism for the chemotactic response of endothelial cells to tumor angiogenesis factor, *M.A.J. Math. Appl. Med. Biol.,* **10** (1993), 149-168.

[10] H.A. Levine and B.D. Sleeman and M. Nilsen-Hamilton, Mathematical modeling of onset of capillary formation initiating angogenesis, *J. Math. Biol.* **42** (2001), 1432-1466.

[11] D.A.J. Lauffenburger and C.L. Stokes, Analysis of the roles of micro vessles endothelial cell random motility and chemo taxis in angiogenesis, *J. Theo. Biol.,* **152** (1991) 377-403.

[12] S.R. Mc Dougall, A.R.A. Anderson, M.A.J. Chaplain and J.A. Sheratt, Mathematical modeling of flow through vascular networks: Implications for tumor-induced angiogenesis and chemotherapy strategies, *Bull. Math. Biol.,* **64** (2002), 673-702.

[13] V.R. Muthukkaruppen, L. Kubai and R. Auerbach. Tumor-induced neovascularization in mouse eye, *J. Natl. Cancer Inst.,* **69** (1982) 699-705.

[14] M.R. Owen, H.M. Byrne and C. E. Lewis, Mathematical modeling of the use of macrophages as vehicles for drug delivery to hypoxic tumor sites, *J. Thio. Biol.* **226** (2004), 377-391.

[15] G. Pettet, C.P. Please, M. J. Tindall and D. McElwain, The migration of cells in multicell tumor spheroid, *Bull. Math. Biol.,* **03** (2001), 231-257.

[16] M.J. Plank, B.D. Sleeman, and P.F. Jones, A mathematical model of tumor angiogenesis, regulated by vascular endothelial growth factor and the angiopoitins, *J. Theo. Biol.,* **229** (2004), 435-454.

[17] G. Tarabolette and R. Giavazzi, Modeling approaches for angiogenesis, *Europ. J. Cancer,* **40** (2004), 881-889.

Jñānābha, Vol. 38, 2008

DIFFUSION-REACTION MODELFOR MASS TRANSPORTATION IN BRAIN TISSUES

By

Madhu Jain, G.C. Sharma
Department of Mathematics
Institute of Basic Science, Khandari, Agra-282002, Uttar Pradesh, India
E-mail :madhujain@sancharnet.in, gokulchandra@bsnl.in
and
Ram Singh
Department of Mathematics,
St. John's College, Agra, 282002, Uttar Pradesh, India
E-mail: nannu_1982@yahoo.co.in

(Received : April 12, 2008*)*

ABSTRACT

This investigation deals with the study of mass transport in brain tissues. The tissues in the human body are considered as porous medium by keeping the fact in mind that these are made up of dispersed cells seperated by connective voids through which nutrients and other menerals are available to each cell in the tissues. The general diffusion-reaction equations have been used to elucidate the developed bi-layer model. The mass concentrations have been obtained at both layers. The importance of porosity and tortuosity factors has also been discussed. The diffusion-reaction process is more realistic one .

2000 Mathematics Subject Classification : Primary 92C18; Secondary 92C37.
Keywords : Brain tissues, Mass transport, Diffusion-reaction model, Porosity, Tortuosity,

1. Introduction. Diffusion of molecules in the brain and other tissues is important in wide range of biological processes and measurements ranging from the delivery of drugs to diffusion- weighted magnetic resonance imaging. The diffusion molecules may have different diffusion co-efficients and concentrations in the different domains, namely within the tubes' inner core, membrane and within the outer medium. Biological tissues are multi compartmental heterogeneous media composed of cellular and subcellular domains. Diffusion of mass transport is very sensitive to the local environment in tissues, and is affected by the packing geometry of the cells and thier membrane permeability that controls the exchange of the nutrients molecules across the membranes.

The main aim of our investigation is to find out the concentrations of the nutrients at two different layers of the brain tissues by using diffusion-reaction

model. The diffusion-reaction in a geometrically complicated environment is , on a microscopic level, an extremely difficult process. But in biological application, we are often satisfied with macroscopic level. Several researchers have studied diffusion-reaction models in different frameworks.

Vafai and Tien [14] studied a numerical scheme to evaluate the velocity and temperature fields inside a porous medium near an impermeable boundary. They presented a new concept of momentum boundary leyer central to numerical routine. Puri et al. [10] developed a mathematical model to predict the steady state transport of a conservative, neutrally bouyant tracer injected along the centerline into a fully developed turbulent pipe flow. A mathematical model in which, steady state transport of a decaying contaminant in a fractured porous rock matrix by two dimensional diffusion and vertical advection which is treated by Fourier Sine Transform technique, has been studied by Fogden et al. [4]. Tompson and Dougherty [13] studied a two-step particle-in-cell model for reactive mass transport problems in subsurface porous formation and then applied this model to non-lineat diffusion-reaction system. Kangle *et al.* [7] presented a general analytical solution for one-dimensional solute transport in heterogeneous porous media with scale dependent dispersion. McDougall *et al.* [8] used flow modeling tools and tecniques in the field of petroleum engineering. They examined the effects of fluid viscosity, blood vessels size and theoretical networks geometry upon (i) the rate of flow through network (ii) the amount of fluid present in the complete network (iii) the amount of fluid reaching the tumor.

A mathematical model for the oxygen transport in the brain microcirculation in the presence of blood substitues has been developed by Sharan and Popel [12]. Bertuzzi *et al.* [12] investigated a mathematical model for the evolution of a tumor cord after treatment by using extensive numerical simulation. A mathematical model combined with a physical model to simulate the growth characteristics of a single bubble in liquid by the process of rectified diffusion has been developed by Meidani and Hasan [9]. Their model is besed on the coupled momentum energy and mass transport equations. Water diffusion model within the structure of a brain extracellular space for various diffusion parameters of brain tissue namely extracellular space, porosity and tortuosity has numerically been analyzed by Vafai *et al.* [15], Hrabe *et al.* [5] studied a mathematical model for effective diffusion and tortuosity in the extracellular space of the brain . They used a volume-averaging procedure to obtain a general expression for the tortusity in a complex envirnment. Jain and Sharma [6] developed a time dependent mathematical model for oxygen transport in peripheral nerve tissues by using Krogh cylinder symmetry. Sen and Basser [11] presented a mathematical model for diffusion of white matter in brain by using diffusion tensor imaging method to

characterize neuronal tissue in the human brain. Arifin *et al.* [1] used mathematical modeling and simluation to provide a comprehensive review of drug release from polymeric microspheres and of drug transport in adjacent tissue. An experimental investigation of the effect fo water diffusion exchange between compartments on the paramagnetic relaxation enhancement of paramagnetic agent compartment has been presented by Zhang *et al.* [16]. El-Kabeir *et al.* [3] studied a mathematical model for combined magneto-hydrodynamic (MHD) heat and mass transport of non-Darcy natural convection about an impermeable horizontal cylinder in a non Newtonian power law fluid embedded in porous medium under magnetic field and thermal radiation effects.

In this investigation, we construct diffusion-reaction equations to understand the concept of mass transport in brain tissues. We shell develop a mathematical model by considering solute concentration at two different layers. The rest of the paper is arranged as follows. The model description, notations and governing equations have been provided in the Section 2. Section 3 is to devote the analysis of the model. Finally, the conclusions are drawn in the Section 4.

2. Model Description. The present investigation is concerned with the mass transport in porous medium by using diffusion-reaction equations in biological tissues particularly in brain tissues. Two concepts porosity and toruosity have been incorporated in the model. Porosity determines what percentage of the total tissue volume is accessible to the diffusing molecules and on the other hand tortuosity describes the average hindrance of a complex medium relative to obstacles-free medium. One-dimensional transient diffusion-reaction equations have been constructed to find out mass concentration at different situations. It is also assumed that the chemical reaction coefficient at both layers is constant and equal.

Followings are the notations used to formulate the model mathematically:

$C_i(x,t)$	Concentration of mass transport in i^{th} ($i=1,2$) layer
D_i	Diffusivity of i^{th} layer, $i=1,2$
λ_i	Tortouosity of i^{th} layer, $i=1,2$
S_i	Mass source density of i^{th} layer , $i=1,2$
ε_i	Porosity of i^{th} layer, $i=1,2$
K	Chemical reaction co-efficient
t	Time.

Governing Equations. The equations governing the mass transport in the brain tissues due to diffusion-reaction process are constructed as follows:

Then, the deffusion-reaction equation for the concentration in layer-1 is

$$\frac{\partial C_1(x,t)}{\partial t}=\frac{D_1}{\lambda_1^2}\frac{\partial C_1(x,t)}{\partial x^2}-KC_1(x,t)+\frac{S_1}{\varepsilon_1}, \qquad ...(1)$$

The diffusion-reaction equation for the concentration in layer-2 is

$$\frac{\partial C_2(x,t)}{\partial t}=\frac{D_2}{\lambda_2^2}\frac{\partial C_2(x,t)}{\partial x^2}-KC_2(x,t)+\frac{S_2}{\varepsilon_2}, \qquad ...(3)$$

The interface between two layers is at $x=a$. The perfect contacting is assumed at interface and concentration at that position is same.

The initial and boundary conditions for equations (2)-(3) are given as

$$\left.\begin{array}{ll} C_1(x,t)=C_0(x) & \text{at } t=0,\quad 0\le x\le a \\ C_2(x,t)=C_0(x) & \text{at } t=0,\quad 0\le x\le b \\ \dfrac{\partial C_1(x,t)}{\partial t}=0 & \text{at } x=0,\quad t>0 \\ \dfrac{\partial C_2(x,t)}{\partial t}=0 & \text{at } x=b,\quad t>0 \end{array}\right\} \qquad ...(4)$$

The matching conditions are

$$\left.\begin{array}{ll} C_1(x,t)=C_2(x,t) & \text{at } x=a,\quad t>0 \\ D_1\dfrac{\partial C_1(x,t)}{\partial t}=D_2\dfrac{\partial C_2(x,t)}{\partial t} & \text{at } x=a,\quad t>0 \\ D_1=D_2, \dfrac{S_1}{\varepsilon_1}=\dfrac{S_2}{\varepsilon_2}, \lambda_1=\lambda_2 & \text{at } x=a,\quad t>0 \end{array}\right\} \qquad ...(5)$$

3. Mathematical Analysis: Taking Laplace Transform of the equation (2), we obtain

$$\frac{d^2\overline{C_1}(x,p)}{dx^2}-\alpha_1^2\overline{C_1}(x,p)=Y_1 \qquad ...(6)$$

where $\alpha_1^2=\dfrac{\lambda_1^2}{D_1}(K+p)$ and $Y_1=\dfrac{\lambda_1^2}{D_1}\left(C_0-\dfrac{S_1}{\varepsilon_1 p}\right)$.

There fore, $\overline{C_1}(x,p)=Ae^{\alpha_1 x}+Be^{-\alpha_1 x}-\dfrac{Y_1}{\alpha_1^2}$. $\qquad ...(7)$

Again talking the Laplace Transform of the equation (3), we get

$$\frac{d^2\overline{C_2}(x,p)}{dx^2}-\alpha_2^2\overline{C_2}(x,p)=Y_2 \qquad ...(8)$$

where $\alpha_2^2=\dfrac{\lambda_2^2}{D_2}(K+p)$ and $Y_2=\dfrac{\lambda_2^2}{D_2}\left(C_0-\dfrac{S_2}{\varepsilon_2 p}\right)$.

Therefore, $\overline{C}_2(x,p) = Ce^{\alpha_2 x} + De^{-\alpha_2 x} - Y_2/\alpha_2^2$ . ...(9)

Transformed boundary and machining conditions are

$$\left.\begin{array}{ll} \dfrac{d\overline{C}_1(x,p)}{dx} = 0 & \text{at } x = 0, \quad t > 0 \\ \dfrac{d\overline{C}_2(x,p)}{dx} = 0 & \text{at } x = b, \quad t > 0 \\ \overline{C_1}(x,p) = \overline{C}_2(x,p) & \text{at } x = a, \quad t > 0 \\ D_1 \dfrac{d\overline{C_1}(x,p)}{dx} = D_2 \dfrac{d\overline{C}_2(x,p)}{dx} & \text{at } x = a, \quad t > 0 \end{array}\right\} \qquad \text{...(10)}$$

Using these conditions, the solutions of eqs (7) and (9), are:

$$\overline{C}_1(x,p) = \left[\frac{\sum\limits_{m=0}^{\infty} \sum\limits_{n=0}^{\infty} \sum\limits_{r=0}^{n} \sum\limits_{r_1=0}^{n} (-1)^{m+n+r-r_1} \binom{n}{n-r} \binom{n}{n-r_1} \left(\frac{D_1}{D_2}\right)^{n/2} \left(\frac{\lambda_1}{\lambda_2}\right)^{n} \left(\frac{S_1}{\varepsilon_1} - \frac{S_2}{\varepsilon_2}\right)}{\sum\limits_{r_2=0}^{n} \sum\limits_{r_3=0}^{n} \binom{n}{n-r_2} \binom{n}{n-r_1}} \right.$$

$$\left. \frac{\left(\dfrac{e^{-\psi_1\left(\sqrt{K+p}\right)}}{p^2} + \dfrac{e^{-\psi_2\left(\sqrt{K+p}\right)}}{p^2} \right)}{} \right] - \frac{Y_1}{\alpha_1^2} \qquad \text{...(11)}$$

$$\overline{C}_2(x,p) = \left[\frac{\sum\limits_{m=0}^{\infty} \sum\limits_{n=0}^{\infty} \sum\limits_{r=0}^{n} \sum\limits_{r_1=0}^{n} (-1)^{m+n+r_1-r_2} \binom{n}{n-r} \binom{n}{n-r_1} \left(\frac{D_2}{D_1}\right)^{n/2} \left(\frac{\lambda_2}{\lambda_1}\right)^{n} \left(\frac{S_2}{\varepsilon_2} - \frac{S_1}{\varepsilon_1}\right)}{\sum\limits_{r_2=0}^{n} \sum\limits_{r_3=0}^{n} \binom{n}{n-r_2} \binom{n}{n-r_1}} \right.$$

$$\left. \frac{\left(\dfrac{e^{-\psi_3\left(\sqrt{K+p}\right)}}{p^2} + \dfrac{e^{-\psi_4\left(\sqrt{K+p}\right)}}{p^2} \right)}{} \right] - \frac{Y_2}{\alpha_2^2} \qquad \text{...(12)}$$

Taking inverse Laplace Transform of eqs (11) and (12), we get

$$c_1(x,t)=\left[\frac{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r_1=0}^{n}\sum_{r_2=0}^{n}(-1)^{m+n+r-r_3}\binom{n}{n-r_1}\binom{n}{n-r_2}\left(\frac{D_1}{D_2}\right)^{\frac{n}{2}}\left(\frac{\lambda_1}{\lambda_2}\right)^{n}\left(\frac{S_1}{\varepsilon_1}-\frac{S_2}{\varepsilon_2}\right)}{\sum_{r_2=0}^{n}\sum_{r_3=0}^{n}\binom{n}{n-r_2}\binom{n}{n-r_3}}\right.$$

$$\left.\left(\int_0^t\frac{\psi_1}{2\sqrt{\pi u^3}}e^{-K\left(u+\frac{K}{4u}\right)}(t-u)du+\int_0^t\frac{\psi_2}{2\sqrt{\pi u^3}}e^{-K\left(u+\frac{K}{4u}\right)}(t-u)du\right)\right] \quad \text{...(13)}$$

$$-C_0e^{-Kt}+\frac{S_1}{\varepsilon_1K}\left(1-e^{-Kt}\right)$$

$$c_2(x,t)=\left[\frac{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{n}\sum_{r_1=0}^{n}(-1)^{m+n+r_1-r_2}\binom{n}{n-r}\binom{n}{n-r_1}\left(\frac{D_2}{D_1}\right)^{n/2}\left(\frac{\lambda_2}{\lambda_1}\right)^{n}\left(\frac{S_2}{\varepsilon_2}-\frac{S_1}{\varepsilon_1}\right)}{\sum_{r_2=0}^{n}\sum_{r_3=0}^{n}\binom{n}{n-r_2}\binom{n}{n-r_1}}\right.$$

$$\left.\left(\int_0^1\frac{\psi_3}{2\sqrt{\pi u^3}}e^{-K\left(u+\frac{K}{4u}\right)}(t-u)du+\int_0^t\frac{\psi_4}{2\sqrt{\pi u^3}}e^{-K\left(u+\frac{K}{4u}\right)}(t-u)du+\right)\right] \quad (14)$$

$$-C_0e^{-Kt}+\frac{S_2}{\varepsilon_2K}\left(1-e^{-Kt}\right)$$

where

$$\psi_1=\left[\frac{\lambda_1}{\sqrt{D_1}}\{a(2m+2r-2r_2+1)-x\}+\frac{2\lambda_2}{\sqrt{D_2}}\{(b-a)(r_1-r_3)\}\right]\geq 0.$$

$$\psi_2=\left[\frac{\lambda_1}{\sqrt{D_1}}\{a(2m+2r-2r_2+1)+x\}+\frac{2\lambda_2}{\sqrt{D_2}}\{(b-a)(r_1-r_3)\}\right]\geq 0.$$

$$\psi_3 = \left[\frac{\lambda_2}{\sqrt{D_2}} \{2(b-a)(m+r_1-r_3+1)+x-a\} + \frac{2\lambda_1}{\sqrt{D_1}} a r_2 \right] \geq 0.$$

$$\psi_4 = \left[\frac{\lambda_2}{\sqrt{D_2}} \{2(b-a)(m+r_1-r_3+1)-x+a\} + \frac{2\lambda_1}{\sqrt{D_1}} a r_2 \right] \geq 0.$$

4. Conclusion. In this study we have developed a diffusion-recation model for the mass transport in biological tissues, pariticularly in the brain tissues. The transporation of nutrients, oxygen, glucose etc. in the brain tissues from vascular system through diffusion-reaction process has been investigated. The analytical expressions for mass concentration in two layers have been obtained. It has also been obserbed that the porosity and tortuosity affect the mass transport significantly. Our investigation may be helpful in the treatment for brain tumor mentally ratarded patients, in particular when clemical reaction is taken in to consideration.

REFERENCES

[1] D.Y. Arifin, L.Y. Lee, and C.H. Wang, Mathematical modeling and simulation of drug release from microspheres: Implication to drug delivery systems, *Adv. Drug Del. Rev*., **58**, **No**. **12-13** (2006) 1274-1325.

[2] A Bertuzzi, A. D'onofrio, A.Fasano and A.Gandoffi, Regression and re-growth of tumour cords following single-dose anti-cancer treatment, *Bull Math. Bio*.,**65**, No.5.(2003), 903-931.

[3] S.M.M. El-Kabier, M.A. El-Hakim and A.M. Rashad, Group method analysis of combined heat and mass transfer by *MHD* non-Darcy non-Newtonion natural convection adjacent to horizontal cylinder in a saturated porous medium, Appl. Math. Model., 2007 (In Press).

[4] A. Fogden, K.A. Landman, and L.R. White, Contaminant transport in fractured porous media: steady state solutions by Fourier Sine Transform method, *Appl. Math. Model*.,**13**, No. **3** (1989), 160-177.

[5] J. Hrabe, S. Hrabetova and K. Segeth, A model of effective diffusion and tortuosity in the extracellular space of the brain, *Biophy*., **87** (2004), 1606-1617.

[6] M. Jain and G.C. Sharma, A computational solution of mathematical model for oxygen transport in peripheral nerve, *Comput. Bio. Med*., **34** (2004), 633-645.

[7] H. Kangle, T.V. Genuchten and Z. Renduo, Exact solutions for one dimensional transport with asymptotic scale dependent dispersion, *App. Math. Mod*., **20** No.4 (1996), 298-308.

[8] S.R. MeDougall, A.R.A. Anderson, M.A.J. Chaplain and J.A. Sherratt, Mathematical modeling of flow through vascular networks: Implication for tumor-induced angiogenesis and chemotherapy strategies, *Bull. Math. Bio*., **64, No. 4** (2002), 673-702.

[9] A.R.N. Meidani and M. Hasan, Mathematical and physical model of growth due to ultrasound, *App. Math. Mod*., **28 No.4** (2004), 333-351.

[10] A.N. Puri, C.Y. Kou and R.S. Chapman, Turbulent diffusion of mass in circular pipe flow, *App. Math. Mod*., **7, Issue 2** (1983), 135-138.

[11] P.N. Sen, and P.J. Basser, A model for diffusion in white matter in the brain, *Biophy*., **89** (2005), 2927-2938.

[12] M. Sharan, A.S. Popoel, A compartmental model for oxygen transport in brain microculation in the presence of blood substitutes, *J. Theo. Bio.*, **216** (2002), 479-500.

[13] A. F. B. Tompson and D.E. Dougherty, Particle-grid method for reacting flows in porous media with application to Fisher's equation, *App. Math. Mod.*, **16** Issue**7** (1992), 374-383.

[14] K. Vafai and C.L. Tien, Boundary and inertia effect on flow and heat transfer in porous media, *Int. J.Heat Mass Tranhsfer*, **24** (1980), 195-203.

[15] K. Vafai, K. Khanafer and A. Kangarlu, Computational modeling of cerebral diffusion-application of stroke imaging, *Meg. Res. Img.*, **21 Issue 6** (2003), 651-661.

[16] H. Zhang, Y. Xie and T. Ji, Water diffusion-exchange effect on the paramagnetic relaxation enhancement in off-resonance rotating frame, *J. Mag. Res.*, **186 Issue 2** (2007), 259-272.

Jñānābha, Vol. 38, 2008

RELIABILITY ESTIMATION OF PARALLEL-SERIES SYSTEM USING *C-H-A* ALGORITHM

By

A.K. Agarwal

Department of Mathematics

Hindu College, Moradabad, Uttar Pradesh, India

and

Suneet Saxena

Department of Mathematics

Bhagwat Institute of Technology, Muzaffanagar, Uttar Pradesh, India

E-mail: Sunsax75@yahoo.co.in

(Received : March 15, 2008*)*

ABSTRACT

C-H-A algorithm is based on based on minimal path set to evaluate system structure function. Once one obtain the expression for the structure function, the system reliability computation becomes straight-forward. In this paper, we have studied *C-H-A* algorithm and applied it to estimate the reliability of parallel-series system.

2000 Mathematics Subject Classification: Primary Secondary.

Keywords: *C-H-A*-algorithm, structure function, Reliability, Parallel-series system.

1. Introduction. A very general alternative approach for analyzing the reliability of complex system is through the use of the system structure function. Once one obtains the expression for the structure function, the system reliability computation becomes straight-forward. Such attempts have been made in the classical 1975 book by Barlow and Proshan [3]. Various algorithms have been developed to evaluate structure function. Aven Algorithm [1] is based on minimal cut sets. It depends on the initial choices of 2 parameters. Recently Chaudhari, Hu and Afshar [4] proposed new algorithm based on minimal path set to evaluate system structure function. They named it *C-H-A* algorithm.

In this paper, we study *C-H-A* algorithm and apply it to estimate the reliability of parallel- series system.

2. Notations and Definitions

(2.1) Notations

n : Number of components.

x_1 : State of i^{th} component.

x : $(x_1, x_2, ..., x_n)$ states of the components.

$\phi(x)$: Structure function.

P_1 : Probability$\{x_1=1\}$: reliability of i^{th} component.

R : Reliability of system.

(2.2) Definitions

(2.2.1) Structure Function

Let $x_1 = \begin{cases} 1 & \textit{if component } i \textit{ operates} \\ 0 & \textit{if component } i \textit{ has failed.} \end{cases}$

Then the system structure function is defined as

$$\phi(x) = \begin{cases} 1 & \textit{if System operates} \\ 0 & \textit{if System has failed.} \end{cases}$$

(2.2.2) Reliability of System

Reiability of system in terms of structure function is defined as

R=probability $\{\phi(x)=1\}$

$= E\{\phi(x)\}$

where $E\{\phi(x)\} = 0.$ probability $\{\phi(x)\} = 0. + 1.$ Probability $\{(x) = 1\}$.

(2.2.3) Coherent System

A system is coherent when a component reliability improvement does not degrade the system reliability. A coherent system has a structure function that is monotonically increasing.

if $y_i \geq x_i$, for i=1 to n.

then $\phi(y) \geq \phi(x)$.

(2.2.4) Relevant Component

If the inequality is strict for a given component i, then that component is said to be relevant.

(2.2.5) Path Set

A path set is a set of components whose functioning ensures that the system functions.

(2.2.6) Minimal Path set

A minimal path is one in which all the components within the set must function for the system to function.

(2.2.7) (Cut Set)

A set is a cut of components whose failure will result in a system failure.

(2.2.8) Minimal Cut Set

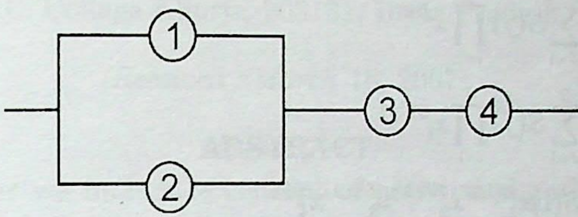
A minimal cut is one in which all the components must fail in order for the

system to fail.

(2.2.9) OR Operaton (Binary)

$1.1=1, \quad 1.0=1, \quad 0.1=1, \quad 0.0=0.$

3. **Reliability of system Using *C-H-A* Algorithm.** Let us consider two components in parallel with two components in series.



Step 1. Find out minimal path sets

Minimal path sets are : {1,3,4}, {2,3,4}.

Step 2. Construct matrix P using minimal path sets. Each column of P represent minimal path. Assign 1 for the component present in path set and 0 for the component not present in path set.

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Steps 3. Construct the design matrix D using the columns of P matrix. Start with two columns of P matrix and apply OR operation on a respective rows and resultant column will be appended in P matrix. Perform same operation on all remaining columns. Process will be extended to three columns, four columns and so on. Once process will stop will obtain design matrix D.

$$D = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Step 4. Construct a vector S whose number of columns are same that of design matrix D. First m elements are 1's, where m is number of columns in P matrix. Next elements(1 or -1) are determined according to rule $(-1)^{i-1}$, where i is the number of columns of P that are taken at a time to be OR' ed in particular step.

$S=[1 \ 1 \ -1]$.

Step 5. Let m be the number of columns in P matrix. Construct the structure function of the system.

$$\phi(x) = \sum_{j=1}^{2^m-1} S(j) \prod_{i=1}^{n} x_i^{D(i,j)}$$

where, $D(i,j)$= element (i,j) of D

$S(j)$ =element j of S

Here, $m=2$, $n=4$.

Structure function $\phi(x)$ is

$$\phi(x)=\sum_{J=1}^{2^2-1} S(j).\prod_{i=1}^{4} x_i^{D(i,j)}$$

$$=\sum_{J=1}^{3} S(j).\prod_{i=1}^{4} x_i^{D(i,j)}$$

$$=S(1)x_1^{D(1,1)}x_2^{D(2,1)}x_3^{D(3,1)}x_4^{D(4,1)}$$

$$+S(2)x_1^{D(1,2)}x_2^{D(2,2)}x_3^{D(3,2)}x_4^{D(4,2)}$$

$$+S(3)x_1^{D(1,3)}x_2^{D(2,3)}x_3^{D(3,3)}x_4^{D(4,3)}$$

$$=1.x_1^1x_2^0x_3^1x_4^1+1.x_1^0x_2^1x_3^1x_4^1-1.x_1^1x_2^1x_3^1x_4^1$$

$$\phi(x)=x_1x_3x_4+x_2x_3x_4-x_1x_2x_3x_4.$$

Reliability is

$$R=E\{\phi(x)\}$$

$$=E(x_1x_3x_4)+E(x_2x_3x_4)-E(x_1x_2x_3x_4)$$

$$=P_1P_3P_4+P_2P_3P_4-P_1P_2P_3P_4.$$

$$R=\{1-(1-p_1)(1-P_2)\}P_3P_4.$$

4. **Disussion.** For last four decades, various methods have been developed to evaluate reliabiality of system. The *C-H-A* method has got its own importance. Being an algorithmic approach, one can write computer program and use computer to evaluate reliability of complex system. Further, the important relibility measures such as Birnbaum Reliability - Importance, Chaudhari bounds can also be evaluated easily.

REFERENCES

[1] T. Aven, Reliability/availability of inherent system based on minimal cut sets, *Reliability Engineering*, **13** (1986), 93-104.

[2] E. Balaguruswami, *Reliability Engineering*, Tata McGraw-Hill Publishing Co. Ltd., New Delhi (2003).

[3] R.E. Barlow and F. Prokshan, *Statistical Theory of Reliability and Testing*, Holt, Winston Richart (1975).

[4] Gopal Chaudhari, Hu, K. and A. Nadar, A new approach to system reliability, *IEEE Trans. on Reliability*, **50**.No **1** (2001), 75-84.

[5] E. Charles Ebeling, *Reliability and Maintainability Engineering*, Tata McGraw-Hill, (2000).

Jñānābha, Vol. 38, 2008

A NOTE ON PAIRWISE SLIGHTLY SEMI-CONTINUOUS FUNCTIONS

By

M.C.Sharma and V.K.Solanki
Department of Mathematics
N.R.E.C. Collage Khurja, 203131, Uttar Pradesh, India

(Received : March 15, 2007)

ABSTRACT

In this paper we introduce concept of pairwise slightly semi-cintinuous function in bitopological spaces and discuss some of the basic properties of them. Several examples are provided of illustrate behaviour of these new classes of functions

2000 Mathematics Subject Classification : 54E55.

Keywords and Phrases : (i,j) clopen set, pairwise slightly continuous, pairwise slightly semi-continuous, pairwise almost semi-continuous, pairwise semi θ-continuous, pairwise weakly semi-continuous, pairwise s- closed, pairwise ultra regular.

1. Introduction. J.C. Kelly [5] initiated the systematic study of bitopological spaces. A set equiped with two topologies is called bitopological spaces. Continuity play an important role in topological and bitopological spaces. In 1980, R.C. Jain [4] introduced the concept of slightly continuity in topological spaces. Recently T.M. Nour [10] defined a slightly semi-continuous functions as a generalization of slightly continuous function using semi-open sets and investigated its properties. In 2000, T. Noiri and G.I. Chae [9] introduce a note on slightly semi-continuous functions in topological spaces.

The object of the present paper is to introduce a new class of function called pairwise slightly semi-continuous functions. This class contained the class of pairwise continuous functions and that of pairwise semi continuous function. Relation between this class and other class of pairwise continuous functions are obtained.

Throughout the present paper the spaces X and Y always represent bitopological spaces (X,P_1,P_2) and (Y,Q_1,Q_2) on which no seperation axioms are assumed. Let $S \subset X$. Then S is said to be **(i, j) semi-open** [8] if $S \subset P_j - Cl$ (P_i-Int(S)) (where P_j–$Cl(S)$ denoted the closure operator with respect to topology P_j and P_i–Int(S) denoted the interior operator with respect to topology P_i, (i, j=1,2, $i \neq j$) and its complement is called **(i, j) semi-closed.** The intersection of all (i, j) semi-closed sets containing S is called the **(i, j) semi-closure** of S and it will be

denoted by $(i, j)\ s\ Cl(S)$. A subset S is said to be (i, j) **semi-regular** if S is both (i, j) semi-open and (i, j) semi closed. A subset S is said to **(i, j) semi θ-open** if S is the union of (i, j) semi-regular sets and the complement of a (i, j) semi θ-open set is called (i, j) **semi θ- closed**. A subset S is said to be **(i, j) clopen** if S is P_i–closed and P_j–open set in X.

In this note we will denote the family of all (i, j) semi-open (resp. P_i–open, (i, j) semi-regular and (i, j) clopen of (X,P_1,P_2) by $(i, j)SO(X)$(resp. P_i-open(X),(i, j) $SR(X)$ and $(i, j)CO(X)$), and denote the family of (i, j) semi-open (resp. P_i-open, (i, j) semi-regular and (i, j) clopen) set of (X,P_1,P_2) containing x by $(i, j)SO(X,x)$(resp. P_i (X,x), $(i, j)SR(X,x)$ and $(i, j)CO(X,x)$. $i, j=1,2$. $i{\neq}j$.

2. Preliminaries.

Definition 2.1. A function $f{:}(X,P_1,P_2){\rightarrow}(Y,Q_1,Q_2)$ is said to be pairwise-semi continuous [8] ($p.s.C$) resp. pairwise almost semi continuous ($p.a.s.C$)[12] pairwise semi θ-continuous ($p.s.\theta.C.$) [12] and pairwise weakly semi continuous ($p.w.s.C.$) [12]) if for each $x \in X$ and for each $V \in Q_i(y, f(x))$ there exists $U \in (i, j)SO(X,x)$ such that $f(U) \subset V$(resp. $f(U) \subset Q_i - \mathrm{int}(Q_j - Cl(V))$ $f(i, j)sCl(U) \subset Q_j - Cl(V)$ and $f(U) \subset Q_j - Cl(V)$).

Definition 2.2. A function $f{:}(X,P_1,P_2){\rightarrow}(Y,Q_1,Q_2)$ is called to be pairwise almost continuous ($p.a.C$) [2] (resp. pairwise θ-continuous ($p.\theta.C.$)[1], pairwise weakly continuous ($p.w.C$)[2] if for each $x{\in}X$ and for each $V{\in}Q_i$–$(Y, f(x))$, there is $U{\in}P_i(X,x)$ such that $f(U) \subset Q_i$–Int$(Q_j$–$Cl(V))$ (resp. $f(P_j$–$Cl(U) \subset Q_j$–$Cl(V)$, $f(U) \subset Q_j$–$Cl(V)$).

Definition 2.3. [11] A function $f{:}(X,P_1,P_2){\rightarrow}(Y,Q_1,Q_2)$ is called slightly semi-continuous ($p.sl.s.C.$)(resp. pairwise slightly continuous ($p.sl.C$) if for each $x{\in}X$ and for each $V{\in}(i, j)CO(Y, f(x))$, there exists $U{\in}(i, j)SO(X,x)$ (resp. $U{\in}P_i(X,x)$) such that $f(U) \subset V$, $i, j=1,2$ and $i{\neq}j$.

A function $f{:}(X,P_1,P_2){\rightarrow}(Y,Q_1,Q_2)$ is said to be pairwise slightly semi-continuous (resp. pairwise slightly continuous) if inverse image of each (i, j)-clopen set of Y is (i, j) semi-open (resp. P_i–open) in X; $i, j=1,2$, $i{\neq}j$.

The following diagram is obtained :

$$\begin{array}{ccccccccc}
p.C & \Rightarrow & p.a.C & \Rightarrow & p.\theta.C. & \Rightarrow & p.w.C & \Rightarrow & p.sl.C \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
p.s.C & \Rightarrow & p.a.s.C & \Rightarrow & p.\theta.s.C & \Rightarrow & p.w.s.C & \Rightarrow & p.sl.s.C
\end{array}$$

diagram 1

Remark 2.4. *It was point out in* [11] *that pairwise slightly continuity implies pairwise slighltly semi-continuity, but not conversely. Its counter examples are not given in it.*

Example 2.5. Let $X=\{a, b, c\}$, $P_1=\{\phi, X, \{a\},\{b\},\{a,b\}\}$, $P_2=\{\phi, X, \{a\},\{a,b\}\}$ and let $Q_1=\{\phi, X, \{a\}\}$, $Q_2=\{\phi, Y, \{b,c\}\}$. Then the mapping $f:(X,P_1,P_2)\to(Y,Q_1,Q_2)$ is pairwise slihtly semi-continuous but not pairwise slightly continuous for $f^{-1}(\{a\})$ is (j, i) semi-open and (i, j) semi-closed, but not P_i –closed in (X,P_1,P_2).

Theorem 2.6. *The pairwise set-connectedness and the pairwise slightly continuity are equivalent for a surjeetive function.*

Proof. A surjection $f:(X,P_1,P_2)\to(Y,Q_1,Q_2)$ is pairwise set-connected if and only if $f^{-1}(F)$ is (i, j) clopen in X for each (i, j) clopen set F of Y. It is easy to prove that a function $f:(X,P_1,P_2)\to(Y,Q_1,Q_2)$ is pairwise slightly continuous if and only if $f^{-1}(F)$ is P_1–open in X for each (i, j) clopen set F of Y. Therefore, the proof is obvious.

Theorem 2.7. For a function $f:(X,P_1,P_2)\to(Y,Q_1,Q_2)$ the following are equivalent:

(a) f is pairwise slightly semi-continuous,

(b) $f^{-1}(V)\in(i, j)SO(X)$ for each $V\in(i, j)CO(Y)$,

(c) $f^{-1}(V)$ is (j, i) semi-open and (i, j) semi-closed for each $V\in(i, j)CO(Y)$.

3. Properties of pairwise slightly semi-continuity.

Theorem 3.1. The following are equivalent for a function $f:(X,P_1,P_2)\to(Y,Q_1,Q_2)$:

(a) f is pairwise slightly semi-continuous,

(b) For each $x\in X$ and for each $(V)\in(i, j)CO(Y,f(x))$, there exists $U\in(i, j)$ $SR(X,x)$ such that $f(U)\subset V$,

(c) For each $x\in X$ and for each $(V)\in(i, j)CO(Y, f(x))$, there is $U\in(i, j)$ $SO(X,x)$ such that $f((i, j)sCl(U))\subset V$.

Proof. (a)$\Rightarrow$(b). Let $x\in X$ and $V\in(i, j)CO(Y, f(x))$. By Theorem 2.7, we have $f^{-1}(V)\in$ $(i,j)SR(X,x)$. Put $U=f^{-1}(V))$, then $x\in U$ and $f(U)\subset V$.

(b)$\Rightarrow$(c). It is obvious and is thus omitted.

(c)$\Rightarrow$(a). If $U\in(i, j)SO(X)$, then $(i, j)sCl(U)\in(i, j)SO(X)$.

Definition 3.2. *A bitopological space (X,P_1,P_2) is called*

(a) **Pairwise semi-T_2**[6] (resp. pairwise ultra Hausdorff or pairwise UT_2) if for each pair of distinct points x, y of X, there exists a P_1–semi-open (resp. P_1–clopen) set U and a P_2–semi-open (resp. P_2–clopen) set V such that $x\in U$, $y\in V$ and $U\cap V=\phi$.

(b) **Pairwise s-normal**[7] (resp.pairwise ultra normal) if for every P_i-closed set A and P_j-closed set B such that $A\cap B=\phi$, there exist $U\in SO(X)$ (resp $Co(X,P_j)$ and $V\in SO(X,P_i)$(resp.$CO(X,P_i)$such that $A\subset U$, $B\subset V$ and $U\cap V=\phi$,where $i, j=1,2$, $i\neq j$

(c) **Pairwise s-closed** (resp.pairwise mildly compact) if every (i,j) semi regular (resp.(i, j) clopen) cover of (X,P_1,P_2) has a finite subcover. $i, j=1,2$ and $i\neq j$.

Theorem 3.3. If $f:(X,P_1,P_2)\to(Y,Q_1,Q_2)$ is a pairwise slightly semi-continuous injection and Y is pairwise UT_2, then X is pairwise semi-T_2.

Proof. Let $x_1,x_2 \in X$ and $x_1 \neq x_2$. Then since f is injective and Y is pairwise UT_2, $f(x_1) \neq f(x_2)$ and there exist, $V_1,V_2 \in (i,j)CO(Y)$ such that $f(x_1) \in V_1$, $f(x_2) \in V_2$ and $V_1 \cap V_2 = \phi$. By Theorem 2.7, $x_i \in f^{-1}(V_i) \in (i,j)SO(X)$ for $i=1,2$ and $f^{-1}(V_1) \cap f^{-1}(V_2) = \phi$. Thus X is pairwise semi-T_2.

Theorem 3.4. If $f:(X,P_1,P_2) \to (Y,Q_1,Q_2)$ is a pairwise slightly semi-continuous, P_2-closed injection and Y is pairwise ultra normal, then X is pairwise s-normal.

Proof. Let F_1 and F_2 be disjoint (P_1,P_2)-closed subsets of X. Since f is P_2-closed and injective, $f(F_1)$ and $f(F_2)$ are disjoint (Q_1,Q_2)-closed subsets of Y. Since Y is pairwise ultra normal, $f(F_1)$ and $f(F_2)$ are separeted by disjoint P_i–clopen sets V_1 and P_j-clopen V_2, respectively. Hence $F_i \subset f^{-1}(V_i)$, $f^{-1}(V_i) \in (i,j)SO(X)$ for $i=1,2$ from Theorem 2.7 and $f^{-1}(V_1) \cap f^{-1}(V_2) = \phi$. Thus X is pairwise s-normal.

Theorem 3.5. If $f:(X,P_1,P_2) \to (Y,Q_1,Q_2)$ is a pairwise slightly semi-continuous surjection and (X,P_1,P_2) is pairwise s-closed, then Y is pairwise mildly compact.

Proof. Let $\{V_\alpha \mid V_\alpha \in (i,j)CO(Y), \alpha \in \nabla\}$ be a cover of Y. Since f is pairwise slightly semi-continuous, by the Theorem 2.7 $\{f^{-1}(V_\alpha) \mid \alpha \in \nabla\}$ is a (i,j) semi-regular cover of X and so there is a finite subset ∇_0 of ∇ such that $X = \bigcup_{\alpha \in \nabla_0} f^{-1}(V_\alpha)$.

Therefore,

$$Y = \bigcup_{\alpha \in \nabla_0} V_\alpha$$

since f is surjective. Thus Y is pairwise mildly compact.

Theorem 3.6 If $f:(X,P_1,P_2) \to (Y,Q_1,Q_2)$ is pairwise slightly semi-continuous and Y is pairwise UT_2, then the graph $G(f)$ of f is (i,j) semi θ-closed in the bitopological product space $X \times Y$.

Proof. Let $(x,y) \notin G(f)$, then $y \neq f(x)$. Since Y is pairwise UT_2, there exist $V \in (i,j)$ $CO(Y,y)$ and $W \in (i,j)$ $CO(Y, f(X))$ such that $V \cap W = \phi$. Since f is pairwise slightly semi-continuous, by the Theorem 2.7 there exist $U \in (i,j)$ $SR(X,x)$ and $V \in (i,j)$ $CO(Y,y)$, $(x,y) \in U \times V$ and $U \times V \in (i,j)SR(X \times Y)$. Hence $G(f)$ is (i,j) semi θ- closed.

Theorem 3.7. If $f:(X,P_1,P_2) \to (Y,Q_1,Q_2)$ is pairwise slightly semi-continuous and (Y,Q_1,Q_2) is pairwise UT_2, then $A = \{(X_1,X_2) / f(x_1) = f(x_2)\}$ is)(i,j)semi -θ closed in the bitopological product space $X \times X$.

Proof. Let $(X_1,X_2) \notin A$. Then $f(x_1) \neq f(x_2)$. Since Y is pairwise UT_2, there exist $V_1 \in (i,j)$ $CO(Y, f(X_1))$ and $V_2 \in (i,j)$ $CO(Y, f(X_2))$ such that $V_1 \cap V_2 = \phi$. Since f is pairwise slightly semi-continuous, there exist $U_1, U_2 \in (i,j)SR(X)$ such that $X_1 \in U_1$ and $f(U_1) \subset V_1$ for $i=1,2$. Therefore, $(X_1,X_2) \in U_1 \times U_2$, $U_1 \times U_2 \in (i,j)$ $SR(X \times X)$, and $(U_1 \times U_2) \cap A = \phi$. So A is (i,j) semi θ-closed in bitopological product space $X \times X$.

Definition 3.8. [3] A bitopological(X,P_1,P_2) is said to be pairwise extremally

disconnected if P_2 -closure of each P_1-open set of (X, P_1, P_2) is P_1-open

Lemma 3.9. Let (X, P_1, P_2) be pairwise exteremally disconnected space, then $U\in (i, j)SR(X)$ if and only if $U\in (i, j)CO(X), i,j=1,2$ and $i\neq j$.

Proof. Let $U\in(i, j)SR(X)$. Since $U\in (i, j)SO(X), P_j\text{-}Cl\ (U)=P_j\text{-}Cl(P_i\text{-Int}\ (U)$ and so $P_i\text{-}Cl(U)\in P_i(X)$. Since U is (i, j) semi-closed, $P_i\text{-Int}(U) = U = P_j\text{-}Cl\ (U)$ and hence U is *(i, j)* clopen. The convers is obvious.

Theorem 3.10. If $f:(X,P_1,P_2)\rightarrow(Y,Q_1,Q_2)$ is pairwise slightly semi-continuous, and X is pairwise extremally disconnected then f is pairwise slightly continuous.

Proof. Let $x\in X$ and $V\in(i, j)CO(Y,f(x))$. Since f is pairwise slightly semi-continuous by Theorem 2.7, there exists $U\in (i, j)SR(X,x)$ such that $f(U)\subset V$, since X is pairwise extremally disconnected by the Lemma 3.9, $U\in (i, j)\ CO\ (X)$ and hence f is pairwise slightly continous.

Difinition 3.11. A function $f: (X, P_1, P_2)\rightarrow(Y, Q_1, Q_2)$ is called pairwise almost strongly θ-semi continuous *(p,a.θ.s.C.)* if for each x $\in$X and for each $V\in Q_i(Y,f(x))$, there exists $U\in (i, j)SO(X,x)$ such that $f((i, j)\ sCl(U))\subset (i, j)sCl(V)$ (resp.$f((i, j)sCl(U))\subset V$).

Theorem 3.12. If $f:(X,P_1,P_2)\rightarrow(Y,Q_1,Q_2)$ is pairwise slightly semi-continuous and (Y, Q_1, Q_2) is pairwise extremally disconnected, then f is pairwise almost strogly θ-semi-continuous.

Proof. Let $x\in X$ and $V\in Q_i(Y,f(x))$, then $(i, j)\ sCl(V) = Q_i\text{-}Int(Q_j\text{-}Cl(V))$ is (i, j) regular open in (Y, Q_1, Q_2). Since Y is pairwise extremally disconnected, $(i, j)\ sCl(V)\in (i, j)\ CO(Y)$. Since f is pairwise slightly semi-continuous, by Theorem 3.1, there exists $U\in (i, j)\ SO(X, x)$ such that $f((i, j)sCl(U)\subset(i, j)sCl(V))$. So f is pairwise almost strongly θ-semi-continuous.

Corollary3.13. [11] If $f:(X,P_1,P_2)\rightarrow(Y,Q_1,Q_2)$ is pairwise slightly semi-continuous and (Y, Q_1, Q_2) is pairwise extremally disconnected. Then f is pairwise weakly semi-continuous.

Difinition 3.14. A bitopological space (X, P_1, P_2) is called pairwise ultra regular if for each $U\in P_i(X)$ and for each $x\in U$, there exists $O\in(i, j)\ CO\ (X)$ such that $x\in O\subset U$.

Theorem 3.15. If f : $(X, P_1, P_2)\rightarrow(Y, Q_1, Q_2)$ is pairwise slightly semi-continuous and (Y,Q_1,Q_2) is pairwise ultra regular, then f is pairwise stroghly θ-semi-continuous.

Proof. Let $x\in X$ and $V\in Q_i(Y,f(X))$. Since (Y,Q_1,Q_2) is pairwise ultra regular, there is $W\in(i,j)CO(Y)$ such that $f(x)\in W\subset V$. Since f is pairwise slightly semi-continuous, by the Theorem 3.1 there is $U\subset(i,j)SO(X,x)$ such that $f((i, j)sCl(U))\subset W$ and so $f(i,j)s\dot{C}l(U))\subset V$. Thus f is strongly θ-semi-continuous.

We have to the following diagram:

$p.\delta.C$

$$\begin{array}{ccccccccc}
\Downarrow & & & & & & & & \\
P.C & \Rightarrow & P.a.C & \Rightarrow & P.\theta.C & \Rightarrow & P.w.C & \Rightarrow & P.sl.C \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
P.s.C & \Rightarrow & P.a.s.C & \Rightarrow & P.s.\theta.C & \Rightarrow & P.w.s.C & \Rightarrow & P.sl.s.C \\
\Uparrow & & \Uparrow & & & & & & \\
P.st.\theta.s.C & \Rightarrow & P.st.\theta.\underline{s.C} & & & & & &
\end{array}$$

REFERENCES

[1] S.Bose, Almost open sets, almost closed, θ-continuous and almost quasi compact mapping in bitopological spaces, *Bull, Cal. Math.Soc;* **73** (1981),345-354.

[2] S.Bose and D. Sinha, Pairwise almost continuous map and weakly continuous map in bitopological spaces, *Bull, Cal. Math.Soc.,* **74** (1982), 195-206.

[3] M.C.Datta, *J.Austr. Math. Soc*, **14**(1972) 119-128.

[4] R.C.Jain, *The Role of Regularly Open Sets in General Topology*, Ph.D. Thisis, Meerut University, Institute of Advanced Studies, Meerut, India(1980).

[5] J.C. Kelly, Bitopological spaces, *Proc. London Math. Soc.,* **13** (1961),71-89.

[6] S.N. Maheshwari and R. Prasad, Some new separation axioms in bitopological spaces, *Mathematica;* **12** (27) (1975), 159-162.

[7] S.N. Maheshwari and R. Prasad, On pairwise s-normal sapces, *Kyungpook Math. J.,* **15** (1975), 37-40.

[8] S.N. Maheshwari and R. Prasad, On pairwise irresolute functions, *Mathematica*, **18** (1976), 169-172.

[9] T.Noiri and G.I.Chae, A note on slightly semi-continuous functions, *Bull. Cal. Math. Soc.,* **92** (2)(2000), 87-92.

[10] T.M.Nour,Slightly semi-continuous functions, *Bull. Cal. Math. Soc,* **87** (1995), 187-190.

[11] M.C. Sharma and V.K.Solanki, Pairwise slightly semi-continuous function in bitopological spaces. [communicated]

[12] V.K.Solanki, Almost semi-continuous mapping in bitopological spaces .*The Mathematics Education* **XLII**, **No. 3** (To appear)

Jñānābha, Vol. 38, 2008

FIXED POINT THEOREMS FOR AN ADMISSIBLE CLASS OF ASYMPTOTICALLY REGULAR SEMIGROUPS IN L^P- SPACES

By

G.S. Saluja
Department of Mathematics and I.T.,
Government College of Science, Raipur-492010, Chhattisgarh
E-mail: Saluja_1963@rediffmain.com

(Received : June 15, 2008)

ABSTRACT

The aim of this paper is to prove existence of fixed point of an admissible class of asymptotically regular semigroup T_s *in* L^P-space $(1 < p \leq 2)$ satisfying the condition:

$$\|T_s x - T_s y\|^2 \leq a_s\|x-y\|^2 + b_s(\|x - T_s x\|\,\|y - T_s y\|) + c_s(\|x - T_s y\|\,\|y - T_s x\|), \forall s \in G$$

where a_s, b_s and c_s are non-negative constants satisfying certain conditions. Our result extends and improves the result of Ishihara [10], Ishihara and Taka-hashi [11,12] and many others.

2000 Mathematics subject classification No : 46B20,47H10.

Keywords: Asymptotic regularity, fixed point, L^P-space $(1 < p \leq 2)$

1. Introduction. Let K be a nonempty subset of a Banch space E and $T: K \to K$ be a nonlinear mapping. The mapping T is said to be Lipschitzian if there exists a positive constant k_n such that

$$\|T^n x - T^n y\| \leq k_n \| \|x - y\|,$$

for all $x, y \in K$ and for all $n \in N$. A Lipschitzian mapping is said to be nonexpansive if $k_n = 1$ for all $n \in N$. uniformly k-Lipschitzian if $k_n = k$ for all $n \in N$., and asymptotically nonexpansive if $\lim_{n\to\infty} k_n = 1$, respectively. These mappings were first studied by Goebel and Kirk [7] and Goebel, Kirk and Thele [9]. Lifschitz [14] proved that in a Hillbert space a uniformly k-Lipschitzian mappings with $k < \sqrt{2}$ has a fixed point. Downing and Ray [5] and Ishihara and Takahashi [12] prove that in a Hilbert space a uniformly k-Lipschitzian semigroup with $k < \sqrt{2}$ has a common fixed point. Casini and Maluta [4] and Ishihara and Takahashi [11] proved that a uniformly k-Lipschitzian semigroup in a Banach space E has a common fixed point if $k < N(E)$, where $N(E)$ is the constant of uniformly normal structure.

In these results, the domain of semigroups were assumed to be closed and convex. Ishihara [10] gave the fixed point theorem for Lipschizian semigroups in both Banach and Hilbert spaces in which closedness and convexity of domin were not needed.

The concept of asymptotic regularity is due to Browder and Petryshyn [2]. A mapping $T: E \to E$ is said to be asymptotically regular if

$$\lim_{n\to\infty}\left\|T^{n+1}x - T^n x\right\| = 0,$$

for all $x, y \in E$.

It is well known that if T is nonexpansive then $T_t = tI + (1-t)T$ is asymptotically regular for all $0<t<1$.

Now we consider the following class of mappings, whose n^{th} iterate T^n satisfies the following condition:

$$\left\|T^n x - T^n y\right\|^2 \le a_n\|x-y\|^2 + b_n\left(\left\|x - T^n x\right\|\left\|y - T^n y\right\|\right) + c_n\left(\left\|x - T^n y\right\|.\left\|y - T^n x\right\|\right)$$

for all $x, y \in C$ and $n=1,2,\ldots$ where a_n, b_n and c_n are nonnegative constants satisfying certain conditions. This class of mappings is more general than the class of nonexpansive, asymptotically nonexpansive, Lipschizian and uniformly k-Lipschitzian mappings. The above facts can be seen by taking $b_n = c_n = 0$. The aim of this paper is to prove a fixed point theorem for the above said class of asymptotically regular semigroups in L^p-space $(1 < p \le 2)$. Our result extends and improves the results of Ishihara [10], Ishihara and Takahashi [11,12] and many others.

2. Preliminaries. Let G be a semitopological semigroup, that is, G is a semigroup with a Hausdorff topology such that for each $a \in G$ the mapping $s \to as$ and $s \to sa$ from G to G are continuous. A Semitopological semigroup G is left reversible if any two closed right ideals of G have nonempty intersection. In this case, $(G, \preceq)$ is a directed system when the binary relation" $\preceq$" on G is defined by $a \preceq b$ if and only if $\{a\} \cup \overline{aG} \supseteq \{b\} \cup \overline{bG}$. Examples of left reversible semigroups include commutative and all left amenable semigroups.

Let K be a nonempty subset of a Banach space E. Let $S = \{T_t : t \in G\}$ be a family of mappings from K into itself. Then S is said to be an admissible class of asymptotically regular semigroup on K if it satisfying the following:

(2) for each $x \in K$, the mapping $(s, x) \to T_s(x)$ from $G \times K$ into K is

continuous when $G \times K$ has the product topology,

(3) for each $x \in K$, $h \in G$,

$$\lim_{t\to\infty}\|T_{t+h}x - T_t x\| = 0,$$

(4) for each $s \in G$

(2.0.1) $\|T_s x - T_s y\|^2 \le a_s\|x-y\|^2 + b_s(\|x - T_s x\|\,\|y - T_s y\|) + c_s(\|x - T_s y\|\,\|y - T_s x\|), \forall s \in G$

for all $x, y \in K$, where a_s, b_s and c_s are non-negative constants satisfying certain conditions.

Let $\{B_\alpha : \alpha \in A\}$ be a decreasing net of bounded subsets of a Banach space E. For a nonempty subset K of E, define

$$r\lceil\{B_\alpha\}, x\rceil = \inf_\alpha \sup\{\|x-y\| : y \in B_\alpha\};$$

$$r\lceil\{B_\alpha\}, K\rceil = \inf\{r(\{B_\alpha\}, x) : x \in K\};$$

$$A\lceil\{B_\alpha\}, K\rceil = \{x \in K : r(\{B_\alpha\}, x) = r(\{B_\alpha\}, K)\}.$$

We know that $r\lceil\{B_\alpha\},.\rceil$ is a continuous convex function on E which satisfies the following:

$$|r(\{B_\alpha\}, x) - r(\{B_\alpha\}, y)| \le \|x-y\| \le r(\{B_\alpha\}, x) + r(\{B_\alpha\}, y)$$

for each $x, y \in E$. It is easy to see that E is reflexive and K is closed convex, then $A\lceil\{B_\alpha\}, K\rceil$ is nonempty, and moreover, if E is uniformly convex, then it consists of a single point (cf. [15]).

Let $p > 1$ and denote by λ the number in [0,1] and by $W_p(\lambda)$ the function $\lambda(1-\lambda)^p + \lambda^p(1-\lambda)$.

The functional $\|.\|^p$ is said to be uniformly convex (cf.[25]) on the Banach space E if there exists a positive constant c_p such that for all $\lambda \in [0,1]$ and $x, y \in E$ following inequality holds:

$$\|\lambda x + (1-\lambda)y\|^p \le \lambda\|x\|^p + (1-\lambda)\|y\|^p - W_p(\lambda)c_p\|x-y\|^p.$$

In Hilbert space H, the following equality holds:

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2, \qquad \text{...(**)}$$

for all $x, y \in H$ and $\lambda \in [0,1]$.

If $1 < p \leq 2$, then we have for all x,y in L^P and $\lambda \in [0,1]$, the following inequality holds:

(2.0.2) $\|\lambda x + (1-\lambda)y\|^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)(p-1)\|x-y\|^2$

(The inequality (2.0.2) is contained in [17]).

Xu [24] proved that the functional $\|.\|^p$ is uniformly convex on the whole Banach space E if and only if E is p-uniformly convex, that is, there exists a constant c>0 such that the modulus of convexity (see[8])

$$\delta_E(\in) \geq c \in^p \; for \; all \; 0 \leq \in \leq 2.$$

The normal structure coefficient $N(E)$ *of* E is defined by Bynum [3] as follows:

$N(E) = \inf\left[\frac{diamK}{r_K(K)}\right. K$ is a bounded convex subset of E consisting of more than one point] where

$$\text{diam } K = \sup\{\|x-y\| : x, y \in K\}$$

is the diameter of K and

$$r_K(K) = \inf_{x \in K}\left\{\sup_{y \in K}\|x-y\|\right\}$$

is the Chebyshev radius of K relative to itself.

The space E is said to have uniformly notmal structure if $N(E) > 1$. It is known that a uniformly convex Banach space has uniformly normal structure and for a Hilbert space H,$N(H)= \sqrt{2}$. Recently, Pichugov [18] (cf. Prus [20] calculated that

$$N(L^P) = \min\{2^{1/p}, 2^{(p-1)/p}\}, 1 < p < \infty.$$

Some estimates for normal structure coefficient in other Banach spaces may be found in Prus [21]. For a subset K, we denote by $\overline{co}K$ the closure of the convex hull of K.

3. **Main Result**. In this section, we give our main result:

Theorem 3.1. Let E be a L^{P} space $(1 < p \leq 2)$, K a nonempty subset of E, G a left reversible semitopological semigoup and $S = \{T_t : t \in G\}$ be an admissible class of asymptotically regular semigroup on K satisfying the condition:

$\|T_s x - T_s y\|^2 \leq a_s\|x-y\|^2 + b_s(\|x - T_s x\|\|y - T_s y\|) + c_s(\|x - T_s y\|\|y - T_s x\|), \forall s \in G$ ...(*)

for all $x, y \in k$, where a_s, b_s and c_s are non-negative constants satisfying certain conditions such that

$$\left\{\frac{\gamma+\sqrt{\gamma^2+4(p-1)((\alpha+\gamma)-1)}}{2(p-1)}\cdot\frac{(\alpha+2\gamma)^{\frac{1}{2}}}{N}\right\}<1$$

where

$$\alpha = \limsup_{s\to\infty} a_s, \qquad \gamma = \limsup_{s\to\infty} c_s .$$

Suppose that $\{T_t y : t \in G\}$ is bounded for some $y \in K$ and there exists a closed subset C of K such that $\bigcap_s \overline{co}\{T_t x : t \succeq s\} \subseteq C$ for all $x \in K$. Then there exists a $z \in C$ such that $T_s(z) = z$ for all $s \in G$.

Proof. Let $B_s(x) = \overline{co}\{T_t(x); t \geq s\}$ and let $B(x) = \bigcap_s B_s(x)$ for $s \in G$ and $z \in K$. Define $\{x_n : n \geq 0\}$ by induction as follows:

$$x_0 = y, \qquad x_n = A(\{B_s(x_{n-1})\}, B(x_{n-1})), \text{ for } n \geq 1$$

since $B(x) \subseteq C \subseteq K$ for all $x \in K, \{x_n\}$ is well defined. Let

$$r_m = r(\{B_s(x_m)\}, B(x_m)),$$
$$D_m = r(\{B_s(x_m)\}, B(x_{m-1})), \qquad m \geq 1.$$

Now for each $s, t \in G$ and $x, y \in K$, we have

$$\|T_s T_t x - T_s y\|^2 \leq a_s\|x-y\|^2 + b_s(\|T_t x - T_s T_t x\|.\|y - T_s y\|) + c_s(\|y - T_s T_t x\|.\|T_t x - T_s y\|)$$

and so

$$(3.0.3)\ \|T_s T_t x - T_s y\|^2 \leq a_s\|x-y\|^2 + b_s(\|T_t x - T_s T_t x\|.\|y - T_s y\|) + c_s(\|y - T_t x\| + \|T_t x - T_s T_t x\|).$$

$$(\|T_t x - y\| + \|y - T_s y\|).$$

Then from $x_m \in B(x_{m-1}) = \bigcap_t B_t(x_{m-1})$ and a result of Ishihara and Takahashi [11], we have

$$(3.04) \qquad r_m = r(\{B_s(x_m)\}, B(x_m)) \leq \frac{1}{N} \inf_s diam(B_s(x_m))$$

Now using (3.0.4), we have

$$\inf_s diam(B_s(x_m)) = \inf_s \sup\{\|T_a x_m - T_b x_m\| : a, b \succeq s\}$$

$$\le \limsup_t \left\{ \limsup_s \|T_s x_m - T_t x_m\| \right\}$$

$$\le \limsup_t \left\{ \limsup_s \|T_t T_s x_m - T_t x_m\| \right\}$$

$$\le \limsup_t \Big[\limsup_s \Big(at\|T_s x_m - x_m\|^2 + b_t \big(\|T_s x_m - T_t T_s x_m\| . \|x_m - T_t x_m\|\big)$$

$$+ c_t \big(\|T_s x_m - T_t x_m\| . \|x_m - T_t T_s x_m\|\big)\Big)^{1/2} \Big]$$

$$\le \limsup_t \Big[\limsup_s \Big(at\|T_s x_m - x_m\|^2 + b_t \big(\|T_s x_m - T_t T_s x_m\| . \|x_m - T_t x_m\|\big) + c_t$$

$$\big(\|T_s x_m - x_m\| + \|x_m - T_t x_m\|\big) . \big(\|x_m - T_s x_m\| + \|T_s x_m - T_t T_s x_m\|\big)\Big)^{1/2} \Big]$$

Taking the $\limsup$ as $s \to \infty$ and by asymptotic regularity of T, we get

$$\inf_t diam(B_s(x_m)) \le \limsup_t \left[\alpha D_m^2 + \gamma\left(D_m + \|x_m - T_t x_m\| D_m\right)^{1/2} \right].$$

Again taking the $\limsup$ as $t \to \infty$, we get

$$\inf_s diam(B_s(x_m)) \le (\alpha + 2\gamma)^{1/2} D_m,$$

and hance using (3.0.4), we have

$$r_m \le \frac{(\alpha + 2\gamma)^{1/2}}{N} D_m, \tag{3.0.5}$$

where

$$\limsup_{t\to\infty} a_t = \alpha, \qquad \limsup_{t\to\infty} c_t = \gamma$$

and N is the normal structure coefficient of E.
Again from (2.0.2) and (3.0.3), we have

$$\|\lambda x_{m+1} + (1-\lambda)T_t x_{m+1} - T_s x_m\|^2 \lambda(1-\lambda)(p-1)\|x_{m+1} - T_t x_{m+1}\|^2$$

$$\le \lambda\|x_{m+1} - T_s x_m\|^2 + (1-\lambda)\|T_t x_{m+1} - T_s x_m\|^2$$

$$\le \lambda\|x_{m+1} - T_s x_m\|^2 + (1-\lambda)\|T_t x_{m+1} - T_t T_s x_m\|^2$$

$$\le \lambda\|x_{m+1} - T_s x_m\|^2 + (1-\lambda) \left[a_t\|x_{m+1} - T_s x_m\|^2 + b_t \big(\|x_{m+1} - T_t x_{m+1}\| \|T_s x_m - T_s T_t x_m\|\big) \right.$$

$$+c_t\left(\|x_{m+1}-T_sx_m\|+\|T_sx_m-T_tT_sx_m\|\right)\left(\|T_sx_m-x_{m+1}\|+\|x_{m+1}-T_tx_{m+1}\|\right)\Big].$$

Taking the $\limsup$ as $s\to\infty$ and by asymptotic regularity of T, we have

$$r_m^2+\lambda(1-\lambda)(p-1)\|x_{m+1}-T_tx_{m+1}\|^2\le\lambda r_m^2+(1-\lambda)\left[a_tr_m^2+c_tr_m\left(r_m+\|x_{m+1}-T_tx_{m+1}\|\right)\right].$$

Again taking the $\sup$ as $t\to\infty$, we have

$$r_m^2+\lambda(1-\lambda)(p-1)D_{m+1}^2\le\lambda r_m^2+(1-\lambda)\left|ar_m^2+\gamma r_m(r_m+D_{m+1})\right|$$

or,

$$r_m^2+\frac{\lambda(1-\lambda)}{(1-\lambda)}(p-1)D_{m+1}^2\le\alpha r_m^2+\gamma r_m(r_m+D_{m+1}).$$

Letting $\lambda\to1$, we get

$$r_m^2+(p-1)D_{m+1}^2\le(\alpha+\gamma)r_m^2+\gamma r_mD_{m+1}$$

or,

$$F(t)=(p-1)t^2-\gamma r_mt-((\alpha+\gamma)-1)r_m^2\le0,$$

where $t=D_{m+1}$.

It can be easily seen that

$$F(t)\le0\ \text{for all t}=\frac{\gamma+\sqrt{\gamma^2+4(p-1)((\alpha+\gamma)-1)}}{2(p-1)}r_m.$$

It follows from (3.0.5) that

$$D_{m+1}=\frac{\gamma+\sqrt{\gamma^2+4(p-1)((\alpha+\gamma)-1)}}{2(p-1)}\frac{(\alpha+2\gamma)^{1/2}}{N}D_m.$$

Hence,

$$D_{m+1}\le B.D_m,m\ge1$$

where

$$B=\frac{\gamma+\sqrt{\gamma^2+4(p-1)((\alpha+\gamma)-1)}}{2(p-1)}\frac{(\alpha+2\gamma)^{1/2}}{N}<1,$$

by the assumption of the theorem. Since

$$\begin{aligned}\|x_{m+1}-x_m\|&\le r(\{B_s(x_m)\},x_{m+1})+r(\{B_s(x_m)\},x_m)\\&\le r_m+D_m\\&\le 2D_m\end{aligned}$$

$$\le \ldots$$
$$\le \ldots$$
$$\le 2B^{m-1}D_1 \to 0 \quad \text{as } m \to \infty,$$

it follows that $\{x_m\}$ is a Cauchy sequerce. Let $z = \lim_{m\to\infty} x_m$. Then we have

$$\|z - T_s z\| \le \|z - x_m\| + \|x_m - T_s x_m\| + \|T_s x_m - T_s z\|$$

$$\le \|z - x_m\| + \|x_m - T_s x_m\| + \left[a_s\|x_m - z\|^2 + b_s\left(\|x_m - T_s x_m\|.\|z - T_s z\|\right) + c_s\left(\|x_m - T_s x\|.\|z - T_s x_m\|\right)\right]^{1/2}$$

$$\le \|z - x_m\| + \|x_m - T_s x_m\| + \left[a_s\|x_m - z\|^2 + b_s\left(\|x_m - T_s x_m\|.\|z - T_s z\|\right) + c_s\left(\|x_m - z\| + \|z - T_s z\|\right)\left(\|z - x_m\| + \|x_m - T_s x_m\|\right)\right]^{1/2}$$

$$\le \|z - x_m\| + D_m + \left[a_s\|x_m - z\|^2 + b_s D_m\|z - T_s z\| + c_s\left(\|x_m - z\| + \|z - T_s z\|\right)\left(\|z - x_m\| + D_m\right)\right]^{1/2}$$

Taking the limit as $m \to \infty$ each side, we have

$$\|z - T_s z\| \le \lim_{m\to\infty}\Big[\|z - x_m\| + D_m + (a_s\|x_m - z\|^2 + b_s D_m\|z - T_s z\| +$$
$$c_s\left(\|x_m - z\| + \|z - T_s z\|\right)\left(\|z - x_m\| + D_m\right))^{1/2}\Big] \to 0$$

for all $s \in G$. Hence we have $T_s z = z$ for all $s \in G$. This completes the proof.

As a direct consequence of Theorem 3.1, we have the following results:

Corollary 3.1. Let E be a L^{P}- space $(1 < p \le 2), K$ a nonempty subject of E, and $T : K \to K$ be a self mapping satisfying the condition:

$$\|T^n x - T^n y\|^2 \le a_n\|x - y\|^2 + b_n\left(\|x - T^n x\|.\|y - T^n y\|\right) + c_n\left(\|x - T^n y\|.\|y - T^n x\|\right)$$

for all $x, y \in K$ and $n \ge 1$, where a_n, b_n and c_n are nonnegative constatnt satisfying certain conditions. Suppose that $\{T^n y : n \ge 1\}$ is bounded for some $y \in K$ and there exists a closed subset C of K such that $\bigcap_n \overline{co}\{T^k x : k \ge n\} \subseteq C$ for all $x \in K$. If

$$\frac{\gamma + \sqrt{\gamma^2 4(p-1)((\alpha+\gamma)-1)}}{2(p-1)} \frac{(\alpha + 2\gamma)^{1/2}}{N} < 1,$$

where

$$\alpha = \limsup_{n\to\infty} a_n, \qquad \gamma = \limsup_{n\to\infty} c_n,$$

then there exists a point z in C such that Tz=z.

By Therorem 3.1 and equation (**), we immediately obtain the following:

Theorem 3.2: Let E be a Hillbert space, K a nonempty subset of E, G a left reversible semitopological semigoup and $S = \{T_t : t \in G\}$ be an admissible class of

asymptotically regular semigroup on K satisfying the condition:

$$\|T_s x - T_s y\|^2 \le a_s \|x - y\|^2 + b_s (\|x - T_s x\| \|y - T_s y\|) + c_s (\|x - T_s y\| \|y - T_s x\|), \forall s \in G$$

for all $x, y \in K$, where a_s, b_s and c_s are non-negative constants satisfying certain conditions such that

$$\left\{ \frac{\gamma + \sqrt{\gamma^2 4((\alpha+\gamma)-1)}}{2} \cdot \frac{(\alpha+2\gamma)^{\frac{1}{2}}}{\sqrt{2}} \right\} < 1$$

where $\quad \alpha = \limsup_{s\to\infty} a_s, \qquad \gamma = \limsup_{s\to\infty} c_s,$

Suppose that $\{T_t y : t \in G\}$ is bounded for some $y \in K$ and there exists a closed subset C of K such that $\bigcap_s \overline{co}\{T_t x : t \ge s\} \subseteq C$ for all $x \in K$. Then there exists a $z \in C$ such that $T_s z = z$ for all $s \in G$.

If we put $b_s = c_s = 0$ and $a_s = k_s^2$ in inequality (*) of Theorem 3.1, we get the following results as corollary:

Corollary 3.2 (see[[10], **Theorem1**): Let K be a nonempty subset of a Hillbert space H, G a left reversible semitopological semigroup, and $S = \{T_t : t \in G\}$ a Lipschitzian semigroup on K with $\limsup_{s\to\infty} k_s < \sqrt{2}$. Suppose that $\{T_t y : t \in G\}$ is bounded for some $y \in K$ and there exists a closed subset C of K such that $\bigcap_s \overline{co}\{T_t x : t \ge s\} \subseteq C$ for all $x \in K$. Then there exists a $z \in C$ such that $T_s z = z$ for all $s \in G$.

Remark 1. Theorem 3.2 extends and improves the corresponding results of Ishihara [10], Ishihara and Takahashi [11] and Downing and Ray [5]. The reason is that the above authors have considered Lipschitzian of uniformly Lipschitzian semigroups in Hilbert space whereas we consider asymptotically regular semigroup satisfying the condition (*) which is more general than Lipschitzian or uniformly Lipschitzain semigroups.

Remark 2. Our results also extend several known results given in the literature.

REFERENCES

[1] J. Barros-Neto, *An Introduction to the Theory of Distributions*, Pure and Applied Mathematics, 14, Marcel Dekker, New York, 1973.

[2] F.E. Browder and V.W. Petryshyn, The solution of nonlinear functional equations in Banach spaces, *Bull. Amer. Math. Soc.*, **72**(1966), 571-576.

[3] W.L. Bynum, Normal structure coefficient for Banach space *Pacific J. Math.*, **86** (1980), 427-436.

[4] E. Casini and E. Maluta, Fixed points of unifromly lipschitizian mappings in spaces with uniformly normal structure, *Nonlinear Anal. TMA* **9** (1985), 103-108.

[5] D.J. Dowing and W.O. Ray, Uniformly Lipschitzian semigroups in Hilbert space, *Canad. Math. Bull.*, **25 No. 2** (1982) 210-214.

[6] N. Dunford and J.T. Schwartz, Linear Operators, I. General Theory, *Pure and Applied Mathematics*, Vol **7** Intersciences Publishers, New York (1958).

[7] K. Goebel and W.A. Kirk, A fixed point theorem for transformations whose iterates have unifromly Lipschitz constant, *Studia Math*, **47** (1973), 135-140.

[8] K. Goebel and W.A. Kirk, Topics in metric fixed point theory, *Cambridge Stud. Adv. Math.* **28** Cambridge University Press, London, 1990.

[9] K. Goebel, W.A. Kirk and R.L. Thele, Uniformly Lipschitzian families of transformations in Banach Spaces, *Canad. J. Math.*, **26** (1974), 1245-1256.

[10] H. Ishihara, Fixed point theorems for Lipschitzian semigroups, *Canad. Math. Bull.*, **32 No.1** (1989), 90-97

[11] H. Ishihara and W. Takahashi, Fixed point theorems for uniformly Lipschitzian semigroups in Hilbert spaces, *J. Math. Anal. Appl.* **127** no. 1 (1987) 206-210.

[12] H. Ishihara and W. Takahashi, Modulus of convexity, characteristics of convexity and fixed point theorems, *Kodai Math. J.* **10** no. 2 (1987) 197-208.

[14] E.A. Lifschits, Fixed point theorems for operators in strongly convex spaces (Russian) *Voronez. Gos. Univ. Trudy Mat. Fak.* **16** (1975), 23-28.

[15] T.C. Lim, On asymptotic centers and fixed points of nonexpansive mappings, *Canad. J. Math.*, **32** no. 2 (1980) 421-430.

[16] T.C. Lim, *On some L^P* inqualities in best approximation theory, *J. Math. Anal. Appl.*, **154** (1991), 523-528.

[17] T.C. Lim, H.K. Xu and Z.B. Zu, An L^p inequality and its applications to fixed point theory and approximation theory in progress in Approximation Theory, *P. Neval and A. Pikus* (eds.) Academic Press, New York 1991, 609-624.

[18] S.A. Pichugove, The Jung constant of the space *Lp*, *Mat. Zametki*, **43 No.5** (1988), 604-614, *Translation in Math. Notes*, **43** (1988), 348-354.

[19] B. Prus and R. Smarzewski, Strongly unique best approximations and centers in uniformly convex spaces, *J. Math. Anal. Appl.* **121** no. 1 (1987), 10-21.

[20] B. Prus, On Bynum's fixed point theorems, *Atti. Sem. Mat. Fis. Univ. Modena*, **38 No. 2** (1990), 535-545.

[21] B. Prus, Some estimates for the normal structure coefficients in Banach spaces, *Rend Circ. Mat. Palermo* (2) **40** 1 (1991) 128-135.

[22] R. Smarzewski, Strongly unique best approximation in Banach spaces, *II J. Approx. Theory* **51 No.2** (1987), 202-217.

[23] R. Smarzewski. On an inequalityi of Bynum and Drew, *J. Math. Anal. Appl.* **150** No. 1 (1990), 146-150.

[24] H.K. Xu, Inequality in Banach spaces with applications, *Nonlinear Anal.* **16 No. 12** (1991), 1127-1138.

[25] C. Zalinescu, On uniformly convex function, *J. Math. Anal. Appl.*, **95** (1983), 344-374.

Jñānābha, Vol. 38, 2008

SOME INTEGRAL FORMULAS INVOLVING A GENERAL SEQUENCE OF FUNCTIONS, A GENERAL CLASS OF POLYNOMIALS AND THE MULTIVARIABLE H-FUNCTION

By

V.G. Gupta

Department of Mathematics,
University of Rajasthan, Jaipur-302004, Rajasthan, India

and

Suman Jain

S.S. Jain Subodh Postgraduate College, Jaipur-302004, Rajasthan, Indian

(Received : August 14, 2008*)*

ABSTRACT

In this paper we establish four integral formulas involving a general sequence of functions, a general class of polynomials and the multivariable H-function. The results established here are quite general in nature from which one can derive a large number of (known and new) integrals by specializing the parameters suitably of the various functions involved therein.

2000 Mathematics Subject Classification No : Primerey... secondary

Keywords: General sequence of functions, General class of polynomials. Multivariable H-function.

1. Introduction. Srivastava and Panda [c.f.,e.g., [10], [15], [16] and [17] introduced the multivariable H-function in a series of papers defined and represented in the following contracted notation ([15], p. 130)

$$H[z_1,\ldots,z_r]=H\begin{matrix}0,N:m_1,n_1;\ldots;m_r,n_r\\ p,Q:p_1,q_1;\ldots;p_rq_r\end{matrix}\left[\begin{matrix}z_1\\ \vdots\\ z_r\end{matrix}\left|\begin{matrix}\left(a_j;\alpha_j',\ldots,\alpha_j^{(r)}\right)_{1,p}\left(c_j',\gamma_j'\right)_{1,p_1};\ldots;\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,p_r}\\ \left(b_j;\beta_j',\ldots,\beta_j^{(r)}\right)_{1,Q}\left(d_j',\delta_j'\right)_{1,q_1};\ldots;\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,q_r}\end{matrix}\right.\right] \quad (1.1)$$

to denote the H-function of r complex variables $z_1,\ldots,z_r$. See Srivastava, Gupta and Goyal ([18], p.251, Eq.(4.17)) for details of this function.

A general class of polynomials occurring in this paper was introduced by Srivastava [[20], p.158, eq.(1.1)] defined by

$$S_n^m[x]=\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}x^k, n=0,1,2,\ldots \quad \ldots(1.2)$$

where m is arbitrary positive integer and the coefficients $A_{n,k}(n,k\geq 0)$ are arbitary constants, real or complex.

By suitably specializing the coefficients $A_{n,k}$ the polynomial $S_n^m[x]$ can be reduced to the well known polynomials. These include among others, the Jacobi, Hermite, Legendre, Tchebycheff, Laguerre polynomials.

Agarwal and Chaubey [1], Srivastava and Manocha [21], Salim [13] and several others have studied a general sequence of functions. In the present paper we shall study the following useful series formula for a general sequence of functions

$$R_n^{\alpha,\beta}[x;a,b,c,d;p,q;\gamma,\delta]=\sum_{v,u,t,e,h}\psi(v,u,t,e,h)x^s \qquad ...(1.3)$$

where

$$\sum_{v,u,t,e,h}=\sum_{v=0}^{n}\sum_{u=0}^{v}\sum_{t=0}^{n}\sum_{e=0}^{t}\sum_{h=0}^{\infty}, \left|\frac{c}{d}x^q\right|<1, s=kn+q(h+v)+pt \qquad ...(1.4)$$

and

$$\psi(v,u,t,e,h)=\frac{b^{\gamma m-t}k^n a^t c^{v+h} d^{\delta n-v-h}(-v)_u(-t)_c(\alpha)_t}{k_n v!u!t!e!h!}$$

$$\frac{(-1)^{t+h}(\alpha-\gamma n)_e(-\beta-\delta n)_v(v-\delta n)_h}{(1-\alpha-t)_e}\left(\frac{pe+\lambda+qu}{k}\right)_v \qquad ...(1.5)$$

By suitably specializing the parameters involved in (1.3) a general sequence of function reduced to generalized polynomial set studied by Raizada [11], a class of polynomials introduced by Fujiwara [5], a well known Jacobi polynomials given in Rainville [10] and several others.

2. Main Integrals. In this section we establish four main integrals.

First Integral

$$\int_0^{\infty} t^{\alpha-1/2}(t+a)^{-\alpha}(t+b)^{-\alpha}S_n^m\left[\left(\frac{t}{(t+a)(t+b)}\right)^{\sigma}\right]$$

$$R_D^{A.B}\left[\left(\frac{t}{(t+a)(t+b)}\right)^{\rho}, c', d', f, g; p, q; \gamma, \delta\right]$$

$$H\left[\left\{\frac{t}{(t+a)(t+b)}\right\}^{\rho_1} z_1, \ldots, \left\{\frac{t}{(t+a)(t+b)}\right\}^{\rho_r} z_r\right]dt$$

$$=\sum_{u,v,w,e,h}\sum_{l=0}^{[n/m]}\frac{(-n)_{ml}}{l!}A_{n,l}\,\psi(u,v,w,e,h)\sqrt{\pi}\left(\sqrt{a}+\sqrt{b}\right)^{1-2(\alpha+\sigma l+\rho s)}$$

$$H^{(1)}_{\ell,\alpha,\rho}\left[z_1\left(\sqrt{a}+\sqrt{b}\right)^{-2\rho_1},\ldots,z_r\left(\sqrt{a}+\sqrt{b}\right)^{-2\rho_r}\right] \quad \ldots(2.1)$$

where

$$H^{(1)}_{\ell,\alpha,\rho}[z_1,\ldots,z_r]=H^{0,N+1:m_1,n_1;\ldots;m_r,n_r}_{p+1,Q+1:p_1,q_1;\ldots;p_r q_r}$$

$$\left[\begin{matrix}z_1\\ \vdots\\ z_r\end{matrix}\left|\begin{matrix}\left(\frac{3}{2}-\sigma\ell-\alpha-\rho s;\rho_1,\ldots,\rho_r\right),\left(a_j;\alpha'_j,\ldots,\alpha_j^{(r)}\right)_{1,P}:\left(c'_j,\gamma'_j\right)_{1,p_1};\ldots;\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,p_r}\\ \left(1-\alpha-\sigma\ell-\rho s;\rho_1,\ldots,\rho_r\right),\left(b_j;\beta'_j,\ldots,\beta_j^{(r)}\right)_{1,Q}:\left(d'_j,\delta'_j\right)_{1,q_1};\ldots;\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,q_r}\end{matrix}\right.\right] \quad \ldots(2.2)$$

Provided a,b,σ,ρ and $\rho_1(i=1,\ldots,r)$ are all positive Re $(\alpha)>0$ and

$$\mathrm{Re}\left(\alpha+\frac{1}{2}+\sigma\ell+\rho s\right)+\sum_{i=1}^{r}\rho_i\xi_i>0$$

where

$$\xi_i=\min_{1\le j\le\mu^i}\mathrm{Re}\left[\frac{d_j^{(i)}}{\delta_j^{(i)}}\right]>0 \quad \ldots(2.3)$$

The infinite series occurring on the right hand side of (2.1) converages absolutely. Also $\psi(u,v,w,e,h)$ is given by (1.5).

Second Integral

$$\int_0^t x^{-\alpha-1}(t-x)^{\alpha-1}e^{-\beta/x}S^m_n\left[\left(\frac{t-x}{x}\right)^{\sigma}\right]R_D^{A,B}\left[\left(\frac{t-x}{x}\right);c',d',f,g;p,q;\gamma,\delta\right]$$

$$H\left[\left(\frac{t-x}{x}\right)^{\rho_1}z_1,\ldots,\left(\frac{t-x}{x}\right)^{\rho_r}z_r\right]dx$$

$$=\beta^{-\alpha}t^{\alpha-1}e^{-\beta/t}\sum_{u,v,w,e,h}\sum_{t=0}^{[n/m]}\psi(u,v,w,e,h)\frac{(-n)_{mt}}{\ell!}A_{n,t}\left(\frac{t}{\beta}\right)^{\sigma\ell+\rho s}$$

$$H^{(2)}_{\ell,\alpha,\rho}\left[\left(\frac{t}{\beta}\right)^{\rho_1}z_1,\ldots,\left(\frac{t}{\beta}\right)^{\rho_r}z_r\right] \quad \ldots(2.4)$$

where

$$H^{(2)}_{\ell,\alpha,\rho}[z_1,\ldots,z_r]=H^{0,N+1:m_1,n_1;\ldots;m_r,n_r}_{p+1,Q+1:p_1,q_1;\ldots;p_r q_r}$$

$$\left[\begin{array}{c} z_1 \\ \vdots \\ z_r \end{array}\left|\begin{array}{l} (1-\alpha-\sigma\ell-\rho s;\rho_1,\ldots,\rho_r),(a_j;\alpha_j',\ldots,\alpha_j^{(r)})_{1,P}:(c_j',\gamma_j')_{1,p_1};\ldots;(c_j^{(r)},\gamma_j^{(r)})_{1,p_r} \\ ((b_j;\beta_j',\ldots,\beta_j^{(r)})_{1,Q}:(d_j',\delta_j')_{1,q_1};\ldots;(d_j^{(r)},\delta_j^{(r)})_{1,q_r} \end{array}\right.\right] \quad \ldots(2.5)$$

$$\mathrm{Re}(\alpha)>0, \mathrm{Re}(\beta)<0, \mathrm{Re}(\alpha+\sigma\ell+\rho s)-\sum_{i=1}^{r}\rho_i\xi_i<0$$

Third Integral

$$\int_0^1\int_0^1\left(\frac{1-x}{1-xy}y\right)^{\alpha}\left(\frac{1-y}{1-xy}\right)^{\beta}\frac{1-xy}{(1-x)(1-y)}S_n^m\left[\left(\frac{1-y}{1-xy}\right)^{\sigma}\right]$$

$$R_D^{A,B}\left[\left(\frac{1-y}{1-xy}\right)^{\rho};c',d',f,g;p;\gamma,\delta\right]$$

$$H\left[\left(\frac{1-x}{1-xy}y\right)^{\rho_1}\left(\frac{1-y}{1-xy}\right)^{\sigma_1}z_1,\ldots,\left(\frac{1-x}{1-xy}\right)^{\rho_r}\left(\frac{1-y}{1-xy}\right)^{\sigma_r}z_r\right]dxdy$$

$$=\sum_{u,v,w,e,h}\sum_{\ell=0}^{[n/m]}\frac{(-n)_{m\ell}}{\ell!}A_{n,\ell}\psi(u,v,w,e,h)H_{\ell,\alpha,\beta,\rho}^{(3)}[z_1,z_r,\ldots,z_r] \quad \ldots(2.6)$$

where

$$H_{\ell,\alpha,\beta,\rho}^{(3)}[z_1,\ldots,z_r]=H_{P+2,Q+1:p_1,q_1;\ldots;p_r,q_r}^{0,N+2:m_1,n_1;\ldots;m_r,n_r}$$

$$\left[\begin{array}{c} z_1 \\ \vdots \\ z_r \end{array}\left|\begin{array}{l} (1-\alpha;\rho_1,\ldots,\rho_r),(1-\beta-\sigma\ell-\rho s.\sigma_1,\ldots,\sigma_r),(a_j;\alpha_j',\ldots,\alpha_j^{(r)})_{1,P},(c_j',\gamma_j')_{1,p_1};\ldots;(c_j^{(r)},\gamma_j^{(r)})_{1,p_r} \\ (1-\alpha-\beta-\sigma\ell-\rho s,(\rho_1+\sigma_1),\ldots,(\rho_r+\sigma_r)((b_j;\beta_j',\ldots,\beta_j^{(r)})_{1,Q},(d_j',\delta_j')_{1,q_1};\ldots;(d_j^{(r)},\delta_j^{(r)})_{1,q_r} \end{array}\right.\right], \quad \ldots(2.7)$$

where $\sigma,\rho,\rho_i,\sigma_i>0(i=1,\ldots,r)$

and $\mathrm{Re}(\alpha)+\sum_{i=1}^{r}\rho_i\xi_i>0, \mathrm{Re}(\beta+\sigma\ell+\rho s)+\sum_{i=1}^{r}\sigma_i\xi_i>0$

[ξ_i is given by (2.3)]

Fourth Integral

$$\int_0^1\int_0^1 f(xy)y^{\alpha}(1-x)^{\alpha-1}(1-y)^{\beta-1}S_n^m\left[(1-y)^{\sigma}\right]R_D^{A,B}\left[(1-y)^{\rho};c',d',f,g;p;\gamma,\delta\right]$$

$$H\left[y^{\sigma_1}(1-x)^{\sigma_1}(1-y)^{\rho_1}z_1,\ldots,y^{\sigma_r}(1-x)^{\sigma_r}(1-y)^{\rho_r}z_r\right]dxdy$$

$$=\sum_{u,v,w,e,h}\sum_{\ell=0}^{[n/m]}\frac{(-n)_{m\ell}}{\ell!}A_{n,\ell}\psi(u,v,w,e,h)\int_0^1 H^{(4)}_{\ell,\alpha,\beta,\rho}\left(z_1(1-z)^{\rho_1+\sigma_1},\ldots,z_r(1-z)^{\rho_r+\sigma_r}\right)$$

$$f(z)(1-z)^{\alpha+\beta+\sigma\ell+\rho s+1}dz \qquad \ldots(2.8)$$

where

$$H^{(4)}_{\ell,\alpha,\beta,\rho}\left[z_1(1-z)^{\rho_1+\sigma_1},\ldots,z_r(1-z)^{\rho_r+\sigma_r}\right]$$

$$=H^{0,N+2:m_1,n_1;\ldots;m_r,n_r}_{p+2,Q+1:p_1,q_1;\ldots;p_r q_r}$$

$$\left[\begin{matrix} z_1(1-z)^{\rho_1+\sigma_1} \\ \vdots \\ z_r(1-z)^{\rho_r+\sigma_r}\end{matrix}\left|\begin{matrix}(1-\alpha;\sigma_1,\ldots,\sigma_r),(1-\beta-\ell\sigma-\rho s.\rho_1,\ldots,\rho_r),(a_j;\alpha'_j,\ldots,\alpha^{(r)}_j)_{1,P},(c'_j,\gamma'_j)_{1,p_1};\ldots;(c^{(r)}_j,\gamma^{(r)}_j)_{1,p_r} \\ (1-\alpha-\beta-\ell\sigma-\rho s;(\rho_1+\sigma_1),\ldots,(\rho_r+\sigma_r)),(b_j;\beta'_j,\ldots,\beta^{(r)}_j)_{1,Q},(d'_j,\delta'_j)_{1,q_1};\ldots;(d^{(r)}_j,\delta^{(r)}_j)_{1,q_r}\end{matrix}\right.\right], \qquad \ldots(2.9)$$

Provided that $f(z)$ is so chosen that the integral (2.8) exists, $\sigma,\rho,\rho_i,\sigma_i>0(i=1,\ldots,r)$ and

$$\mathrm{Re}(\alpha)+\sum_{i=1}^{r}\rho_i\xi_i>0,\mathrm{Re}(\beta+\sigma\ell+\rho s)+\sum_{i=1}^{r}\sigma_i\xi_i>0$$

where ξ_i is given by (2.3)

Proof. To establish (2.1) express the general class of polynomial $S_n^m(x)$, general sequence of function $R_D^{A,B}[x,a,b,c,d;p,q;\gamma,\delta]$ on the left hand side of (2.1) by its series (1.2) and (1.3) respectively and the multivariable H-function in terms of Mellin-Barnes type contour integral with the help of (1.1). Change the order of integration and summation (which is easily seen to be justifiable due to the absolute convergence of the integral and sums involved in the process) under the conditions mentioned with (2.1) and then evaluate the resulting t-integral with the help of (2.10) a known result given by Gradshteyn and Rayzhik ([7], p.289, 3, 197(7)).

$$\int_0^\infty x^{\alpha-\frac{1}{2}}(x+\alpha)^{-\alpha}(x+b)^{-\alpha}dx=\sqrt{\pi}\left(\sqrt{a}+\sqrt{b}\right)^{1-2\alpha}\frac{\Gamma\alpha-1/2}{\Gamma\alpha} \qquad \ldots(2.10)$$

[where $\mathrm{Re}(\alpha)>0$]

and then interpreting the result with the help of (1.1) we thus easily arrive at the

right-hand side of (2.1).

The proof of the formulas (2.4), (2.6) and (2.8) are similar to (2.1) in which we use the following results ([7],p. 339, eqn. 3.471(3); [3],p. 145-243)

$$\int_0^t x^{\alpha-1}(t-x)^{\alpha-1}e^{-\beta/x}dx = \Gamma\alpha\beta^{-\alpha}t^{\alpha-1}e^{-\beta/t} \quad ...(2.11)$$

$$\int_0^1\int_0^1 y^{\alpha}(1-x)^{\alpha-1}(1-y)^{\beta-1}(1-xy)^{-(\alpha+\beta-1)}dx\ dy = B(\alpha,\beta) \quad ...(2.12)$$

$$\int_0^1\int_0^1 f(xy)(1-x)^{\alpha-1}(1-y)^{\beta-1}y^{\alpha}dx\ dy$$

$$= B(\alpha,\beta)\int_0^1 f(z)(1-z)^{\alpha+\beta+1}dz \quad ...(2.13)$$

instead of (2.10).

3. Special Cases.

(i) On setting $g=p=1$ and replacing $B = B/\tau, f$ by $-\tau$ and let $k_n^{'} = 1$ in (2.1), we get an integral involving general class of polynomials, generalized polynomial set and the multivariavle H-function

$$\int_0^1 t^{\alpha-\frac{1}{2}}(t+a)^{-\alpha}(t+b)^{-\alpha}S_n^m\left[\left(\frac{t}{(t+a)(t+b)}\right)^{\sigma}\right]$$

$$S_D^{A,B,\tau}\left[\left(\frac{t}{(t+a)(t+b)}\right)^{\rho};q,\gamma,\delta,c',d',\lambda,k\right]$$

$$H\left[\left\{\frac{t}{(t+a)(t+b)}\right\}^{\rho_1}z_1,\ldots,\left\{\frac{t}{(t+a)(t+b)}\right\}^{\rho_r}z_r\right]dt$$

$$= \sum_{u,v,w,e,h}\sum_{\ell=0}^{[n/m]}\frac{(-n)_{m\ell}}{\ell!}A_{n,\ell}\psi(v,u,w,e,h)\sqrt{\pi}\left(\sqrt{a}+\sqrt{b}\right)^{1-2(\alpha+\sigma\ell+\rho s)}$$

$$H_{\ell,\alpha,\rho}^{(1)}\left[z_1\left(\sqrt{a}+\sqrt{b}\right)^{-2\rho_1},\ldots,z_r\left(\sqrt{a}+\sqrt{b}\right)^{-2\rho_r}\right] \quad ...(3.1)$$

The integral (3.1) is valid under same conditions as given for Integral (2.1) and where

$$\psi_1(u,v,w,e,h) = \frac{d'^{\gamma D-w}k^n c'^w(-1)^{w+v}(-v)_u(-w)_e(A)_w}{v!u!w!e!h!}$$

$$\frac{(-A-\gamma D)_e}{(1-A-w)_e}\left(\frac{B}{\tau}-\delta D\right)_v\left(\frac{e+\lambda+qu}{k}\right)_D(u-\delta D)_D(\tau)^{v+D}$$

and $s = kD + q(h+v) + w$.

On applying the same procedure as above in (2.4), (2.6) and (2.8), we can establish three other integrals.

(ii) If we set $D = \gamma = d' = \lambda = c' = 0, k = q = \delta = 1$ and $\tau \to 0$ in our special case (i) then the generalized polynomial set reduces to unity and we arrive at the result obtained by Gupta [8].

(iii) Taking $n = 0, A_{0,0} = 1$ in special case (ii), we arrive at the results obtained by Chandel and Jain ([2], p. 1421, eqn. (2.1), (2.4), (2.5) and (2.6)).

(iv) Also, the result (2.8) is capable of yielding certain double integrals by an appropriate choice of $f(z)$. Let us take

$$f(z) = (z)^{\delta_1 - 1} {}_2F_1(\lambda_1, \lambda_2, \delta_1, z) in (2.13);$$

substituting this value of $f(z)$ and evaluating the Integral thus obtained on the right hand side of (2.8), we get the following integral using a result in Erdelyi et al ([4], p.399, eqn.(4)):

$$\int_0^1 \int_0^1 x^{\delta_1 - 1} y^{\alpha + \delta_1 - 1} (1-x)^{\alpha - 1} (1-y)^{\beta - 1} {}_2F_1(\lambda_1, \lambda_2; \delta_1; xy)$$

$$S_n^m \left[(1-y)^{\sigma}\right] R_D^{A,B} \left[(1-y)^{\rho}; c', d', f, g; p, q; \gamma, \delta\right]$$

$$H\left[y(1-x)^{\sigma_1}(1-y)^{\rho_1} z_1, \ldots, y(1-x)^{\sigma_r}(1-y)^{\rho_r} z_r\right] dxdy$$

$$= \sum_{u,v,w,e,h} \sum_{\ell=0}^{[n/m]} \frac{(-n)_{m\ell}}{\ell!} A_{n,\ell}\, \psi(u,v,w,e,h) \Gamma \delta_1 H^{(5)}_{\ell,\alpha,\beta,\rho,\lambda_1,\lambda_2}[z_1, z_2, \ldots, z_r]$$

where

$$H^{(5)}_{\ell,\alpha,\beta,\rho,\lambda_1,\lambda_2}[z_1, \ldots, z_r] = H^{0,N+3:m_1,n_1;\ldots;m_r,n_r}_{P+3,Q+2:p_1,q_1;\ldots;p_r,q_r}$$

$$\begin{bmatrix} z_1 \\ \vdots \\ z_r \end{bmatrix} \left| \begin{matrix} 1-\alpha; \sigma_1, \ldots, \sigma_r), (1-\beta-\ell\sigma-\rho s.\rho_1, \ldots, \rho_r), \\ (1-\alpha-\beta-\ell\sigma-\rho s+\lambda_1(\rho_1+\sigma_1), \ldots, (\rho_r+\sigma_r), \end{matrix} \right.$$

$$\left| \begin{matrix} (1-\alpha-\beta-\delta_1-\ell\sigma-\rho s+\lambda_1\lambda_2; (\rho_1+\sigma_r), \ldots, (\rho_r+\sigma_r)); (a_j; \alpha_j', \ldots, \alpha_j^{(r)})_{1,P}, (c_j', \gamma_j')_{1,p_1}; \ldots; (c_j^{(r)}, \gamma_j^{(r)})_{1,p_r} \\ 1-\alpha-\beta-\delta_1-\ell\sigma-\rho s+\lambda_2; (\rho_1+\sigma_1), \ldots, (\rho_r+\sigma_r)); (b_j; \beta_j', \ldots, \beta_j^{(r)})_{1,Q} (d_j', \delta_j')_{1,q_1}; \ldots; (d_j^{(r)}, \delta_j^{(r)})_{1,q_r} \end{matrix} \right]$$

provided that $\mathrm{Re}(\delta_1) > 0, \mathrm{Re}(\alpha+\beta+\ell\sigma+\rho s+2) > 0$ and

$$\mathrm{Re}(\delta_1 \alpha + \beta + \ell\sigma + \rho s + 2 - \lambda_1 - \lambda_2) > 0$$

(ξ_i is given by (2.3))

Several other new results can be obtained by specializing the parameters involved in the result (3.1).

REFERENCES

[1] B.D. Agrawal and J.P. Chaubey, Certain derivation of generating relations for generalized polynomials, *Indian J. Appl. Math.,* **10** (1980), 1155-1157 ibid, **11**(1981), 357-359.

[2] R.S. Chandel and U.C. Jain Certain integral formulas for the multivariable H-functions I, *Indian j. Pure Math.,* **13** (1982), 1420-1426.

[3] J. Edwards, A *Treatise on the Integral Calculus*, **Vol II**, Chelsea Publ. Co., New York, (1922).

[4] A. Erdélyi et al. *Tables of Integral Transforms*, **Vol. II**, McGraw-Hill, New York, 1954.

[5] I. Fujiwara, A unified presentation of classical orthogonal polynomials, *Math. Japon,* **11** (1966), 133-148.

[6] S.P. Goyal and T.O. Salim, Multidimensional Fractional integral operators involving a generalized polynomial set, *Kyungpook Math J.,* **38** (1998), 301-315.

[7] I.S. Gradshteyn and I.M. Ryzhik, *Tables of Integrals, Series and Products*, Academic Press, New York, 1965.

[8] V.G. Gupta, Some integral formulas involving the product of the multivariable H-function and a general class of polynomials, *Indian J.* Math., **31**. (1989), 121-129.

[9] S.L. Kalla, S.P. Goyal and R.K. Agrawal, On multiple integral transformations, *Math. Notae,* **28** (1976-77), 15-27.

[10] E. D. Rainville, *Special Functions,* Chelsea Publ. Co., Bronx, New York (1971).

[11] S.K. Raizada, *A Study of Unified Representation of Special Functions of Mathematical Physics and Their Use in Statistical and Boundary Value Problems*, Ph.D. Thesis, Bundelkhand Univ., Jhansi, India (1991).

[12] M. Saigo, S.P. Goyal and S. Saxena, A theorem relating a generalized Weyl fractional integral, Laplace and Verma transforms with applications, *J. Fractional Calculus*, **13** (1998), 43-56.

[13] T.O. Salim, A series formula of generalized class of polynomials associated with Laplace transform and fractional integral operators, *J. Rajasthan Acad. Phy. Sci.,* **1(3)** (2002), 167-176.

[14] H. M. Srivastava and R. Panda, A note on certain results involving a general class of poplynomials, *Boll Un. Mat. Ital.,* **A(5), 16**(1979), 467-474.

[15] H.M Srivastava, and R. Panda, Some bilateral generating functions for a class of generalized hypergeometric polynomials, *J. Reine Angew .Math* **283/284** (1776), 265-274.

Jñānābha, Vol. 38, 2008

MAXIMIZING SURVIVABILITY OF ACYCLIC MULTI-STATE TRANSMISSION NETWORKS (AMTNs)

By

Raju Singh Gaur and Sanjay Chaudhary

Department of Mathematics, Institute of Basic Sciences, Khandari, Agara, Dr. B.R. Ambedkar University, Agra- 282002, Uttar Pradesh, India

e-mail : rajulg_15@yahoo.co.in

(*Received : October* 5, 2007)

ABSTRACT

In this paper, we evaluate the system survivability of acyclic multi-state transmission networks (*AMTNs*) with vulnerable nodes(positions) using the universal generating function (*UGF*) technique. The *AMTN* survivability is defined as the probability that a signal from root node is transmitted each leaf node. The *AMTNs* consist of a number of positions in which multi-state element (*MEs*) capable of receiving and /or sending a signal are allocated. The *MEs* located at each non-leaf positions. The two *MEs* located at first position. The number of *MEs* is not equal to the number of non-leaf positions. The *AMTNs* survivability is defined as the comparison of two networks. All the *MEs* in the network are assumed to be satistically independent. The signal source is located at each network. The number of leaf positions that can only receive a signal and a number of intermediate positions containing *MEs* copable of transmitting the received signal to some other nodes. The signal transmisson is possible only along links between the nodes. The networks are arranged in such a way that no signal leaving a node can return to this node through any sequence of nodes otherwise we can say that cycle exists.

2000 Mathematics Subject Classification : Primary 62N05; Secondary 90B25.

Keywords : Acyclic multi-state transmisson networks, Multi-state, Survivability, Vulnerability, Universal generating function.

1. Introduction. Acyclic multi-state transmission network (*AMTN*) is a generalization of tree structured. The example of *AMTN* is a radio relay station, where the signal source is allocated at root position (node) and receivers allocated at leaf node. The retransmitter-generating signal situted at each station of radio relay stations, which can transmit the signal to next stations. The *AMTNs* consist of a number of positions in which multi-state elements (*MEs*) capable of receiving and /or sending a signal are allocated. Each network has not position where the signal source is allocated. The number of leaf nodes that can only receive a signal and the number of non-leaf nodes have retransmitter-generating signal. The event

that a *ME* is in specific state is a random event. The probability of this event is assumed to be known for each *ME* and for every of its possible state.

This paper presents the comparison of survivability analysis of two networks. The *MEs* located at each non-leaf position, the two *MEs* located at first position. The number of *MEs* is not equal to the number of non-leaf positions. The first network which has two *MEs* located at first position and other non leaf positions of this network has only - one multi-state element and in second network, the *MEs* located at each non-leaf position. If signal transmission is not working condition from root position to each leaf node then the networks fail otherwise whole network is working condition. Malinowski and Preuss [9] discovered that the acyclic multi-state transmission network is a gerneralization of the tree-structured multi state systems and Hwang and Yao [4] described the concept of multi-state linear consecutively connected network and studied by Kossow and Preuss [5] and Zuo and liang [10]. Gaur and Chaudhary [1,2,3] earlier studied for reliability evaluation of acyclic multi-state network. The algorithm used in this research work is referred by [6,7,8]. In this paper, we consider the case when the *MEs* allocated at the same mode are subject to a common cause failure. When a system operates in battle conditions or is affected by acorrosive medium or other hostile environment. The ability of a system to tolerate both the impact of external factors (attack) and internal causes (accidental failures or errors) should be considered. The measure of this ability is referred to as system survivability. The two *MEs* located at the same node in *ATMN* can be destroyed by a single enternal impact. An external factor usually causes failures of group of system elements sharing some common resource (allocated) with in the same protective casing, having the same power source gathered geographically, etc.). Therefore adding more redundant parallel elements will improve system availability but will not be effective from a vulnerability standpoint without sufficient separation between elements.

The paper is organized as follows : Section 2 introduces the model that is the acyclic multi-state transmission networks model. Section 3 is devoted to Survivability of *AMTNs* using universal generating function technique. Section 4 presents the Disussion and Conclusion.

2. The Model. In the model considered here, the *MEs* located at each non-leaf node $C_i (1 \le i \le N - M)$ where C_i is the non-leaf position can have K_i different states and each state *k* has probability $p_{i\lambda_{ik}}$. The signal can transmit from non-leaf node C_i to the nodes belonging to the set $\wedge_i$. The *ME* cannot transmit a signal to any node : $\lambda_{ik} = \phi$ then the condition is total failure and in the case of

operational state $p_{\lambda_{ik}} = \wedge_i$. There are D available *MEs* with different characteristics with probability $p^d_{i\lambda_{ik}}$: A signal can be transmitted by the *ME* located at C_i only if it reaches this node, such that $\sum_{k=1}^{D} p^a_{i\lambda_{ik}} = 1$ where *ME* $d\,(1 \le d \le D)$ located at C_i can have K_i different states. The states of all the *MEs* are independent. The existence of arc $(C_i, C_j) \in E$ means that a signal can be transmitted directly from the node i to node $j.C_i : C_j \in \wedge_i$ if $(C_i, C_j) \in E$ where j>i. The number of leaf nodes M: $\Phi = \{C_{N\text{-}M+1},\ldots,C_N\}$ and N is the total number of nodes in *AMTN* {note that such numbering is always possible in acyclic directed graph}.

The *MEs* located at C_i if some *ME n* can provide connection from C_i to a set of nodes λ_{ik} and *ME m* can not provide this connection, the state corresponding to set λ_{ik} can be defined for both *MEs,* while $p^n_{i\lambda_{ik}} \neq 0$ and $p^m_{i\lambda_{ik}} \neq 0$.

The system survivability S is defined as a probability that a signal generated at the root node C_1 reaches all the M leaf nodes $C_{N\text{-}M+1},\ldots,C_N$.

In general, the resulting polynomial contains 2^M-1 terms. The suggested method can depend on the moderate values of M for solving *AMTNs.* We determine here *UGF* technique, obtain $\widetilde{U}_i(z)$ for each node C_i using operator θ

$$\widetilde{U}_i(z) = \theta\left(\sum_{k=1}^{\widetilde{K}_i} \widetilde{q}_{ik} z^{\widetilde{V}_{ik}}\right) = \alpha \sum_{k=1}^{\widetilde{K}_i} \widetilde{q}_{ik} z^{\widetilde{V}_{ik}} + \beta z^0 .$$

$\hat{U}_i(z)$ (*u*-functions) is obtained, the values $\hat{\upsilon}_{ik}(1),\ldots,\hat{\upsilon}_{ik}(i)$ representing the presence of signal at nodes $C_1,\ldots,C_i$ are not used further for determining $\hat{U}_m(z)$ for any $m>i$. If the signal can not reach any position from C_{i+1} to C_N independently of states of *MEs* located in these positions then in state k, $\upsilon_{ik}(i+1) = \ldots = \upsilon_{ik}(N) = 0$. The only thing one has to know is the sum of probabilities of states in which these paths exit; it means that we replace all the values $\hat{\upsilon}_{ik}(1),\ldots,\hat{\upsilon}_{ik}(i)$ in vector $\hat{\upsilon}_{ik}$ for $\hat{U}_i(z)$ with zeros and collecting the like terms. For solving the $\varphi(\hat{U}_i(z))$, the vector $\hat{\upsilon}_{ik}$, which contain only zero, is removed and collect the like terms in the resulting polynomial. For calculating the *AMTNs* survivability S, first calculate $\widetilde{U}_{i+1}(z) = \Psi\left(\varphi(\hat{U}_i(z)), \widetilde{U}_{i+1}(z)\right)$ and at last determine

the coefficient of the term of $\varphi(\hat{U}_{N-N}(z))$ in which $\hat{\upsilon}_{N-M}(j)=1$ for all $N-M+1 \le j \le N$.

3. Survivability of *AMTNs* Using Universal Generating Function Technique. Let us consider *AMTN* with $N=5$, $M=2$, presented in *Fig*. 1. *u*-function of two *MEs* located at first position C_1 and other *MEs* located at position C_2, C_3 individually and in *Fig*. 2 individual *MEs* located at position C_1, C_2, C_3.

Case-I. Consider, for example the simplest case in which four identical *MEs* should be allocated, here the number of *MEs* is not equal to the number of non-leaf node. When allocated at node C_1, the *MEs* can have four states :

* Total failure: *ME* does not connect node C_1 with any other node (Probability of this state is $p_{1\phi}^1 = p_{1\phi}^2 = p_{1\phi}$)
* *ME* connects C_1 with C_2 (Probability of this state is $p_{1\{2\}}^1 = p_{1\{2\}}^2 = p_{1\{2\}}$)
* *ME* connects C_1 with C_3 (Probability of this state is $P_{1\{3\}}^1 = p_{1\{3\}}^2 = p_{1\{3\}}$)
* *ME* connects C_1 with both C_2 and C_3 (Probability of this state is $P_{1\{2,3\}}^1 = p_{1\{2,3\}}^2 = p_{1\{2,3\}}$).

When allocated at node C_2, the *MEs* can have two states,

* Total failure : ME does not connect node C_2 with any other node (Probability of this state is $p_{2\phi}$).
* ME connects C_2 with C_3 (Probability of this state is $p_{2\{3\}}$) and C_2 with C_5 is $p_{2\{5\}}$, and C_2 with C_3 and C_5 is $p_{2\{3,5\}}$).

When allocated at node C_3, the *MEs* can have two states,

* Total failure : *ME* does not connect node C_3 with any other node (Probability of this state is $p_{3\phi}$)
* *ME* connects C_3 with C_4 is $p_{3\{4\}}$.

The probability that each node survives during the system operation time is α.

Let us suppose that $p_{1\phi} = p_{2\phi}$ there two possible allocations of the *MEs* with in (*Fig*. 1):

(A) Two *MEs* are located in the first position

(B) and other *MEs* located at second and third position.

When both *MEs* are allocated at position C_1 then we have,

$$u_{11}(z)=p^1_{1\phi}z^{00000}+p^1_{1\{2\}}z^{01000}+p^1_{1\{3\}}z^{00100}+p^1_{1\{2,3\}}z^{01100},$$

$$u_{12}(z)=p^2_{1\phi}z^{00000}+p^2_{1\{2\}}z^{01000}+p^2_{1\{3\}}z^{00100}+p^2_{1\{2,3\}}z^{01100},$$

$$u_2(z)=p_{2\phi}z^{00000}+p_{2\{3\}}z^{00100}+p_{2\{5\}}z^{00001}+p_{2\{3,5\}}z^{00101},$$

$$u_3(z)=p_{3\phi}z^{00000}+p_{3\{4\}}z^{00010}.$$

Following AMTN procedure we obtain,

$$\Omega(u_{11}(z),u_{12}(z))=\Omega\Big(p^1_{1\phi}z^{00000}+p^1_{1\{2\}}z^{01000}+p^1_{1\{3\}}z^{00100}+p^1_{1(2,3)}z^{01100},p^2_{1\{2\}}z^{01000}$$
$$+p^2_{1\{3\}}z^{00100}+p^2_{1\{2,3\}}z^{01100}\Big)$$

$$=p^1_{1\phi}p^2_{1\phi}z^{00000}+\left(p^1_{1\phi}p^2_{1\{2\}}+p^1_{1\{2\}}p^2_{1\phi}+p^1_{1\{2\}}p^2_{1\{2\}}\right)z^{01000}+\Big(p^1_{1\phi}p^2_{1\{3\}}+p^1_{1\{3\}}+p^1_{1\{3\}}p^2_{1\phi}$$
$$+p^1_{1\{3\}}p^2_{1\{3\}}\Big)z^{00100}+\Big(p^1_{1\phi}p^2_{1\{2,3\}}+p^1_{1\{2\}}p^2_{1\{3\}}+p^1_{1\{2\}}p^2_{1,\{2,3\}}+p^1_{1\{3\}}p^2_{1\{2\}}+p^1_{1\{3\}}p^2_{1\{2,3\}}+$$
$$p^1_{1\{2,3\}}p^2_{1\phi}+p^1_{1\{2,3\}}p^2_{1\{2\}}+p^1_{1\{2,3\}}p^2_{1\{2,3\}}z^{01100}$$

$$=p^1_{1\phi}p^2_{1\phi}z^{00000}+\left(p^1_{1\phi}p^2_{1\{2\}}+p^1_{1\{2\}}p^2_{1\phi}+p^1_{1\{2\}}p^2_{1\{2\}}\right)z^{01000}+\Big(p^1_{1\phi}p^2_{1\{3\}}+p^1_{1\{3\}}p^2_{1\phi}$$
$$+p^1_{1\{3\}}p^2_{1\{3\}}\Big)z^{001000}+\Big(p^2_{1\{2,3\}}\left(1-p^1_{1\{2\}}-p^1_{1\{3\}}-p^1_{1\{2,3\}}\right)+p^1_{1\{2\}}p^2_{1\{3\}}+p^1_{1\{2\}}p^2_{1\{2,3\}}+$$
$$p^1_{1\{3\}}p^2_{1\{2\}}+p^1_{1\{3\}}p^2_{1\{2,3\}}+p^1_{1\{2,3\}}\left(1-p^2_{1\{2\}}-p^2_{1\{3\}}-p^2_{1\{2,3\}}\right)+p^1_{1\{2,3\}}p^2_{1\{2\}}+p^1_{1\{2,3\}}$$
$$p^2_{1\{3\}}+p^1_{1\{2,3\}}p^2_{1\{2,3\}}\Big)z^{01100},$$

$$=p^1_{1\phi}p^2_{1\phi}z^{00000}+\left(p^1_{1\phi}p^2_{1\{2\}}+p^1_{1\{2\}}p^2_{1\phi}+p^1_{1\{2\}}p^2_{1\{2\}}\right)z^{01000}+\Big(p^1_{1\phi}p^2_{1\{3\}}+p^1_{1\{3\}}p^2_{1\phi}+$$
$$p^1_{1\{3\}}p^2_{1\{3\}}\Big)z^{00100}+\left(p^2_{1\{2,3\}}+p^1_{1\{2\}}p^2_{1\{3\}}+p^1_{1\{3\}}p^2_{1\{2\}}+p^1_{1\{2,3\}}-p^1_{1\{2,3\}}p^2_{1\{2,3\}}\right)z^{01100},$$

$$\widetilde{U}_1(z)=\theta(u_{11}(z),u_{12}(z))=\left(\beta+\alpha p^1_{1\phi}p^2_{1\phi}\right)z^{00000}+\alpha\left(p^1_{1\phi}p^2_{1\{2\}}+p^1_{1\{2\}}+p^1_{1\{2\}}p^2_{1\phi}+p^1_{1\{2\}}p^2_{1\{2\}}\right)z^{01000}$$
$$+\alpha\left(p^1_{1\phi}p^2_{1\{3\}}+p^1_{1\{3\}}p^1_{1\phi}+p^1_{1\{3\}}p^2_{1\{3\}}\right)z^{00100}+\alpha\Big(p^2_{1\{2,3\}}+p^1_{1\{2\}}p^2_{1\{3\}}+$$
$$p^1_{1\{3\}}p^2_{1\{2\}}+p^1_{1\{2,3\}}-p^1_{1\{2,3\}}p^2_{1\{2,3\}}\Big)z^{01100},$$

$$\widetilde{U}_2(z)=\theta(u_2(z))=\left(\beta+\alpha p_{2\phi}\right)z^{00000}+\alpha p_{2\{3\}}z^{00100}+\alpha p_{2\{5\}}z^{00001}+\alpha p_{2\{3,5\}}z^{00101},$$

$$\widetilde{U}_3(z)=\theta(u_3(z))=\left(\beta+\alpha p_{3\phi}\right)z^{00000}+\alpha p_{3\{4\}}z^{00010}.$$

Now, $\hat{U}_1(z) = \widetilde{U}_1(z)$,

$$\varphi\left(\hat{U}_1(z)\right) = \alpha\left(p^1_{1\phi}p^2_{1\{2\}} + p^1_{1\{2\}}p^2_{1\phi} + p^1_{1\{2\}}p^2_{1\{2\}}\right)z^{01000} + \alpha\left(p^1_{1\phi}p^2_{1\{3\}} + p^1_{1\{3\}}p^1_{1\phi} + p^1_{1\{3\}}p^2_{1\{3\}}\right)z^{00100}$$

$$+ \alpha\left(p^2_{1\{2,3\}} + p^1_{1\{2\}}p^2_{1\{3\}} + p^1_{1\{3\}}p^2_{1\{2\}} + p^1_{1\{2,3\}} - p^1_{1\{2,3\}}p^2_{1\{2,3\}}\right)z^{01100}.$$

$$\hat{U}_2(z) = \Psi\left(\varphi\left(\hat{U}_1(z), \hat{U}_2(z)\right)\right)$$

$$= \psi\left(\alpha\left(p^1_{1\phi}p^2_{1\{2\}} + p^1_{1\{2\}}p^2_{1\phi} + p^1_{1\{2\}}p^2_{1\{2\}}\right)z^{01000} + \alpha\left(p^1_{1\phi}p^2_{1\{3\}} + p^1_{1\{3\}}p^1_{1\phi} + p^1_{1\{3\}}p^2_{1\{3\}}\right)z^{00100}\right.$$

$$+ \alpha\left(p^2_{1\{2,3\}} + p^1_{1\{2\}}p^2_{1\{3\}} + p^1_{1\{3\}}p^2_{1\{2\}} + p^1_{1\{2,3\}} - p^1_{1\{2,3\}}p^2_{1\{2,3\}}\right)z^{01100}, \left(\beta + \alpha p_{2\phi}\right)z^{00000} +$$

$$\left.\alpha p_{2\{3\}}z^{00100} + \alpha p_{2\{5\}}z^{00001} + \alpha p_{2\{3,5\}}z^{00101}\right)$$

$$= \left\{\alpha\left(p^1_{1\phi}p^2_{1\{2\}} + p^1_{1\{2\}}p^2_{1\phi} + p^1_{1\{2\}}p^2_{1\{2\}}p^2_{1\{2\}}\right)\left(\beta + \alpha p_{2\phi}\right)\right\}z^{01000} + \left\{\alpha\left(p^1_{1\phi}p^2_{1\{3\}} + p^1_{1\{3\}}p^1_{1\phi}\right.\right.$$

$$\left. + p^1_{1\{3\}}p^2_{1\{3\}}\right)\left(\beta + \alpha p_{2\phi}\right) + \alpha^2\left(p^1_{1\phi}p^2_{1\{3\}} + p^1_{1\{3\}}p^1_{1\phi} + p^1_{1\{3\}}p^2_{1\{3\}}\right)p_{2\{3\}}\Big\}z^{00100} + \Big\{\alpha\Big(p^2_{1\{2,3\}}$$

$$+ p^1_{1\{2\}}p^2_{1\{3\}} + p^1_{1\{3\}}p^2_{1\{2\}} + p^1_{1\{2,3\}} - p^1_{1\{2,3\}}p^2_{1\{2,3\}}\Big)\left(\beta + \alpha p_{2\phi}\right) + \alpha^2\Big(p^2_{1\{2,3\}} + p^1_{1\{2\}}p^2_{1\{3\}} + p^1_{1\{3\}}p^2_{1\{2\}}$$

$$+ p^1_{1\{2,3\}} - p^1_{1\{2,3\}}p^2_{1\{2,3\}}\Big)p_{2\{3\}} + \alpha^2\left(p^1_{1\phi}p^2_{1\{2\}} + p^1_{1\{2\}}p^2_{1\phi} + p^1_{1\{2\}}p^2_{1\{2\}}\right)p_{2\{3\}}\Big\}z^{01100} +$$

$$\left\{\alpha^2\left(p^1_{1\phi}p^2_{1\{2\}} + p^1_{1\{2\}}p^2_{1\phi} + p^1_{1\{2\}}p^2_{1\{2\}}\right)p_{2\{5\}}\right\}z^{01001} + \Big\{\alpha^2\Big(p^2_{1\{2,3\}} + p^1_{1\{2\}}p^2_{1\{3\}} + p^1_{1\{3\}}p^2_{1\{2\}}$$

$$+ p^1_{1\{2,3\}} - p^1_{1\{2,3\}}p^2_{1\{2,3\}}\Big)p_{2\{5\}} + \alpha^2\left(p^1_{1\phi}p^2_{1\{2\}} + p^1_{1\{2\}}p^2_{1\phi} + p^1_{1\{2\}}p^2_{1\{2\}}\right)p_{2\{3,5\}} + \alpha^2\Big(p^2_{1\{2,3\}}$$

$$+ p^1_{1\{2\}}p^2_{1\{3\}} + p^1_{1\{3\}}p^2_{1\{2\}} + p^1_{1\{2,3\}} - p^1_{1\{2,3\}}p^2_{1\{2,3\}}\Big)p_{2\{3,5\}}\Big\}z^{01101}.$$

$$= \left\{\alpha\left(p^1_{1\phi}p^2_{1\{2\}} + p^1_{1\{2\}}p^2_{1\phi} + p^1_{1\{2\}}p^2_{1\{2\}}\right)\left(\beta + \alpha p_{2\phi}\right)\right\}z^{01000} + \Big\{\alpha\beta\Big(p^1_{1\phi}p^2_{1\{3\}} + p^1_{1\{3\}}p^1_{1\phi}$$

$$+ p^1_{1\{3\}}p^2_{1\{3\}}\Big) + \alpha^2\left(p^1_{1\phi}p^2_{1\{3\}} + p^1_{1\{3\}}p^1_{1\phi} + p^1_{1\{3\}}p^2_{1\{3\}}\right)\left(p_{2\phi} + p_{2\{3\}}\right)\Big\}z^{00100} +$$

$$\Big\{\alpha\beta\left(p^2_{1\{2,3\}} + p^1_{1\{2\}}p^2_{1\{3\}} + p^1_{1\{3\}}p^2_{1\{2\}} + p^1_{1\{2,3\}} - p^1_{1\{2,3\}}p^2_{1\{2,3\}}\right) + \alpha^2\Big(p^2_{1[2,3]} + p^1_{1\{2\}}p^2_{1\{3\}}$$

$$+ p^1_{1\{3\}}p^2_{1\{2\}} + p^1_{1\{2,3\}}p^2_{1\{2,3\}}\Big)\left(p_{2\phi} + p_{2\{3\}}\right)\Big\} + \alpha^2\left(p^1_{1\phi}p^2_{1\{2\}} + p^1_{1\{2\}}p^2_{1\phi} + p^1_{1\{2\}}p^2_{1\{2\}}\right)p_{2\{3\}}\Big\}$$

$$z^{01100} + \left\{\alpha^2\left(p^1_{1\phi}p^2_{1\{2\}} + p^1_{1\{2\}}p^2_{1\phi} + p^1_{1\{2\}}p^2_{1\{2\}}\right)p_{2\{5\}}\right\}z^{01001} + \Big\{\alpha^2\Big(p^2_{1\{2,3\}} + p^1_{1\{2\}}p^2_{1\{3\}}$$

$$+ p^1_{1\{3\}}p^2_1\{2\} + p^1_{1\{2,3\}} - p^1_{1\{2,3\}}p^2_{1\{2,3\}}\Big)\left(p_{2\{5\}} + p_{2\{3,5\}}\right)$$

$$+ \alpha^2\left(p^1_{1\phi}p^2_{1\{2\}} + p^1_{1\{2\}}p^2_{1\phi} + p^1_{1\{2\}}p^2_{1\{2\}}\right)p_{2\{3,5\}}\Big\}z^{01101}$$

$$\varphi\big(\hat{U}_2(z)\big)= \big[\alpha\beta\{(p^1_{1\phi}p^2_{1\{3\}} + p^1_{1\{3\}}p^1_{1\phi} + p^1_{1\{3\}}p^2_{1\{3\}}) + (p^2_{1\{2,3\}} + p^1_{1\{2\}}p^2_{1\{3\}} + p^1_{1\{3\}}p^2_{1\{2\}} + p^1_{1\{2,3\}}$$

$$- p^1_{1\{2,3\}}p^2_{1\{2,3\}})\} + \alpha^2\{((p^1_{1\phi}p^2_{1\{3\}} + p^1_{1\{3\}}p^1_{1\phi} + p^1_{1\{3\}}p^2_{1\{3\}}) + ((p^2_{1\{2,3\}} + p^1_{1\{2\}}p^2_{1\{3\}} + p^1_{1\{3\}}p^2_{1\{2\}}$$

$$+ p^1_{1\{2,3\}} - p^1_{1\{2,3\}}p^2_{1\{2,3\}}))(p_{2\phi} + p_{2\{3\}}) + (p^1_{1\phi}p^2_{1\{2\}} + p^1_{1\{2\}}p^2_{1\phi} + p^1_{1\{2\}}p^2_{1\{2\}})p_{2\{3\}}\}\big]\, z^{00100}$$

$$+ \{\alpha^2(p^1_{1\phi}p^2_{1\{2\}} + p^1_{1\{2\}}p^2_{1\phi} + p^1_{1\{2\}}p^2_{1\{2\}})p_{2\{5\}}\}z^{00001} + \{\alpha^2(p^2_{1\{2,3\}} + p^1_{1\{2\}}p^2_{1\{3\}}$$

$$+ p^1_{1\{3\}}p^2_{1\{2\}} + p^1_{1\{2,3\}} - p^1_{1\{2,3\}}p^2_{1\{2,3\}})(p_{2\{5\}} + p_{2\{3,5\}})$$

$$+ \alpha^2(p^1_{1\phi}p^2_{1\{2\}} + p^1_{1\{2\}}p^2_{1\phi} + p^1_{1\{2\}}p^2_{1\{2\}})p_{2\{3,5\}}\}z^{00101}.$$

$$U_3(z) = \Psi\big(\varphi\big(\hat{U}_2(z), \widetilde{U}_3(z)\big)\big)$$

$$= \big[\alpha\beta\{(p^1_{1\phi}p^2_{1\{3\}} + p^1_{1\{3\}}p^1_{1\phi} + p^1_{1\{3\}}p^2_{1\{3\}}) + (p^2_{1\{2,3\}} + p^1_{1\{2\}}p^2_{1\{3\}} + p^1_{1\{3\}}p^2_{1\{2\}} + p^1_{1\{2,3\}} -$$

$$p^1_{1\{2,3\}}p^2_{1\{2,3\}})\} + \alpha^2\{((p^1_{1\phi}p^2_{1\{3\}} + p^1_{1\{3\}}p^1_{1\phi} + p^1_{1\{3\}}p^2_{1\{3\}}) + ((p^2_{1\{2,3\}} + p^1_{1\{2\}}p^2_{1\{3\}} + p^1_{1\{3\}}p^2_{1\{2\}}$$

$$+ p^1_{1\{2,3\}} - p^1_{1\{2,3\}}p^2_{1\{2,3\}}))(p_{2\phi} + p_{2\{3\}}) + (p^1_{1\phi}p^2_{1\{2\}} + p^1_{1\{2\}}p^2_{1\phi} + p^1_{1\{2\}}p^2_{1\{2\}})p_{2\{3\}}\}(\beta + \alpha p_{3\phi})$$

$$z^{00100} + \{\alpha^2(\beta + \alpha p_{3\phi})(p^1_{1\phi}p^2_{1\{2\}} + p^1_{1\{2\}}p^2_{1\phi} + p^1_{1\{2\}}p^2_{1\{2\}})p_{2\{5\}}\}z^{00001} + \{\alpha^2(\beta + \alpha p_{3\phi})$$

$$(p^2_{1\{2,3\}} + p^1_{1\{2\}}p^2_{1\{3\}} + p^1_{1\{3\}}p^2_{1\{2\}} + p^1_{1\{2,3\}} - p^1_{1\{2,3\}}p^2_{1\{2,3\}})(p_{2\{5\}} + p_{2\{3,5\}}) + \alpha^2(\beta + \alpha p_{3\phi})$$

$$(p^1_{1\phi}p^2_{1\{2\}} + p^1_{1\{2\}}p^2_{1\phi} + p^1_{1\{2\}}p^2_{1\{2\}})p_{2\{3,5\}}\}z^{00101} + \{\alpha^3 p_{3\{4\}}(p^2_{1\{2,3\}} + p^1_{1\{2\}}p^2_{1\{3\}} + p^1_{1\{3\}}p^2_{1\{2\}}$$

$$+ p^1_{1\{2,3\}} - p^1_{1\{2,3\}}p^2_{1\{2,3\}})(p_{2\{5\}} + p_{2\{3,5\}}) + \{\alpha^3 p_{3\{4\}}(p^1_{1\phi}p^2_{1\{2\}} + p^1_{1\{2\}}p^2_{1\phi} + p^1_{1\{2\}}p^2_{1\{2\}})p_{2[3,5]}\}$$

$$z^{00111} + \big[\alpha^2\beta p_{3\{4\}}\{(p^1_{1\phi}p^2_{1\{3\}} + p^1_{1\{3\}}p^1_{1\phi} + p^1_{1\{3\}}p^2_{1\{3\}}) + ((p^2_{1\{2,3\}} + p^1_{1\{2\}}p^2_{1\{3\}} + p^1_{1\{3\}}p^2_{1\{2\}}$$

$$+ p^1_{1\{2,3\}} - p^1_{1\{2,3\}}p^2_{1\{2,3\}})\} + \alpha^3 p_{3(4)}\{(p^1_{1\phi}p^2_{1\{3\}} + p^1_{1\{3\}}p^1_{1\phi} + p^1_{1\{3\}}p^2_{1\{3\}}) + ((p^2_{1\{2,3\}}$$

$$+ p^1_{1\{2\}}p^2_{1\{3\}} + p^1_{1\{3\}}p^2_{1\{2\}} + p^1_{1\{2,3\}} - p^1_{1\{2,3\}}p^2_{1\{2,3\}}))(p_{2\phi} + p_{2\{3\}}) + (p^1_{1\phi}p^2_{1\{2\}} + p^1_{1\{2\}}p^2_{1\phi}$$

$$+ p^1_{1\{2\}}p^2_{1\{2\}})p_{2\{3\}}\}\big]z^{00100.}$$

$$= \big[\alpha\beta\{(p^1_{1\phi}p^2_{1\{3\}} + p^1_{1\{3\}}p^1_{1\phi} + p^1_{1\{3\}}p^2_{1\{3\}}) + (p^2_{1\{2,3\}} + p^1_{1\{2\}}p^2_{1\{3\}} + p^1_{1\{3\}}p^2_{1\{2\}} + p^1_{1\{2,3\}} -$$

$$p^1_{1\{2,3\}}p^2_{1\{2,3\}})\} + \alpha^2\{((p^1_{1\phi}p^2_{1\{3\}} + p^1_{1\{3\}}p^1_{1\phi} + p^1_{1\{3\}}p^2_{1\{3\}}) + (p^2_{1\{2,3\}} + p^1_{1\{2\}}p^2_{1\{3\}} + p^1_{1\{3\}}p^2_{1\{2\}}$$

$$+ p^1_{1\{2,3\}} - p^1_{1\{2,3\}}p^2_{1\{2,3\}}))(p_{2\phi} + p_{2\{3\}}) + (p^1_{1\phi}p^2_{1\{2\}} + p^1_{1\{2\}}p^2_{1\phi} + p^1_{1\{2\}}p^2_{1\{2\}})p_{2\{3\}}\}\big]$$

$$\left(\beta+\alpha p_{3\phi}\right)z^{00100}+\left\{\alpha^{2}\left(p_{1\phi}^{1}p_{1\{2\}}^{2}+p_{1\{2\}}^{1}p_{1\phi}^{2}+p_{1\{2\}}^{1}p_{1\{2\}}^{2}+p_{1\{2,3\}}^{2}+p_{1\{2\}}^{1}p_{1\{3\}}^{2}+p_{1\{3\}}^{1}p_{1\{2\}}^{2}+\right.\right.$$

$$\left.\left.+p_{1\{2,3\}}^{1}-p_{1\{2,3\}}^{1}p_{1\{2,3\}}^{2}\right)\left(\beta+\alpha p_{3\phi}\right)\left(p_{2\{5\}}+p_{2\{3,5\}}\right)\right\}z^{00001}+\left\{\alpha^{3}\left(p_{1\{2,3\}}^{2}+p_{1\{2\}}^{1}p_{1\{3\}}^{2}\right.\right.$$

$$\left.+p_{1\{3\}}^{1}p_{1\{2\}}^{2}+p_{1\{2,3\}}^{1}-p_{1\{2,3\}}^{1}p_{1\{2,3\}}^{2}\right)\left(p_{2\{5\}}+p_{2\{3,5\}}\right)p_{3\{4\}}+\alpha^{3}\left(p_{1\phi}^{1}p_{1\{2\}}^{2}+p_{1\{2\}}^{1}p_{1\phi}^{2}+\right.$$

$$\left.\left.p_{1\{2\}}^{1}p_{1\{2\}}^{2}\right)p_{2\{3,5\}}p_{3\{4\}}\right\}z^{00011}+\left[\alpha^{2}\left(p_{1\phi}^{1}p_{1\{3\}}^{2}+p_{1\{3\}}^{1}p_{1\phi}^{1}+p_{1\{3\}}^{1}p_{1\{3\}}^{2}\right)+\left(p_{1\{2,3\}}^{2}\right.\right.$$

$$\left.+p_{1\{2\}}^{1}p_{1\{3\}}^{2}+p_{1\{3\}}^{1}p_{1\{2\}}^{2}+p_{1\{2,3\}}^{1}-p_{1\{2,3\}}^{1}p_{1\{2,3\}}^{2}\right)\left\{\alpha\left(p_{2\phi}+p_{2\{3\}}\right)+\beta\right\}p_{3(4)}+$$

$$\left.a^{3}\left(p_{1\phi}^{1}p_{1\{2\}}^{2}+p_{1\{2\}}^{1}p_{1\phi}^{2}+p_{1\{2\}}^{1}p_{1\{2\}}^{2}\right)p_{2\{3\}}p_{3\{4\}}\right]z^{00010}.$$

$$\varphi\left(\hat{U}_3(z)\right)=\left\{\alpha^{2}\left(p_{1\phi}^{1}p_{1\{2\}}^{2}+p_{1\{2\}}^{1}p_{1\phi}^{2}+p_{1\{2\}}^{1}p_{1\{2,3\}}^{2}+p_{1\{2,3\}}^{2}+p_{1\{2\}}^{1}p_{1\{3\}}^{2}+p_{1\{3\}}^{1}p_{1\{2\}}^{2}+p_{1\{2,3\}}^{1}-\right.\right.$$

$$\left.\left.p_{1\{2,3\}}^{1}p_{1\{2,3\}}^{2}\right)\left(\beta+\alpha p_{3\phi}\right)\left(p_{2\{5\}}+p_{2\{3,5\}}\right)\right\}z^{00001}+\left\{\alpha^{3}\left(p_{1\{2,3\}}^{2}+p_{1\{2\}}^{1}p_{1\{3\}}^{2}+p_{1\{3\}}^{1}p_{1\{2\}}^{2}+\right.\right.$$

$$\left.p_{1\{2,3\}}^{1}-p_{1\{2,3\}}^{1}p_{1\{2,3\}}^{2}\right)\left(p_{2\{5\}}+p_{2\{3,5\}}\right)p_{3\{4\}}+\alpha^{3}\left(p_{1\phi}^{1}p_{1\{2\}}^{2}+p_{1\{2\}}^{1}p_{1\phi}^{2}+p_{1\{2\}}^{1}p_{1\{2\}}^{2}\right)$$

$$\left.p_{2\{3,5\}}p_{3\{4\}}\right\}z^{00011}+\left[\alpha^{2}\left(p_{1\phi}^{1}p_{1\{3\}}^{2}+p_{1\{3\}}^{1}p_{1\phi}^{1}+p_{1\{3\}}^{1}p_{1\{3\}}^{2}\right)+\left(p_{1\{2,3\}}^{2}+p_{1\{2\}}^{1}p_{1\{3\}}^{2}\right.\right.$$

$$\left.+p_{1\{3\}}^{1}p_{1\{2\}}^{2}+p_{1\{2,3\}}^{1}-p_{1\{2,3\}}^{1}p_{1\{2,3\}}^{2}\right)\left\{\alpha\left(p_{2\phi}k+p_{2\{3\}}\right)+\beta\right\}p_{3\{4\}}+\alpha^{3}\left(p_{1\phi}^{1}p_{1\{2\}}^{2}+\right.$$

$$\left.\left.p_{1\{2\}}^{1}p_{1\phi}^{2}+p_{1\{2\}}^{1}p_{1\{2\}}^{2}\right)p_{2\{3\}}p_{3\{4\}}\right]z^{00010}.$$

$$\varphi\left(\hat{U}_3(z)\right)=\left\{\alpha^{2}\left(2p_{1\phi}p_{1\{2\}}+\left(p_{1\{2\}}\right)^{2}+2p_{1\{2,3\}}+2p_{1\{2\}}p_{1\{3\}}-\left(p_{1\{2,3\}}\right)^{2}\left(\beta+\alpha p_{3\phi}\right)\left(p_{2\{5\}}+p_{2\{3,5\}}\right)\right)\right\}$$

$$z^{00001}+\left\{\alpha^{3}\left(2p_{1\{2,3\}}+2p_{1\{2\}}p_{1\{3\}}-\left(p_{1\{2,3\}}\right)^{2}\right)\left(p_{2\{5\}}+p_{2\{3,5\}}\right)p_{3\{4\}}+\alpha^{3}\left(2p_{1\phi}p_{1\{2\}}+\right.\right.$$

$$\left.\left.\left(p_{1\{2\}}\right)^{2}\right)p_{2\{3,5\}}p_{3\{4\}}\right\}z^{00011}+\left[\alpha^{2}\left(2p_{1\phi}p_{1\{3\}}+\left(p_{1\{3\}}\right)^{2}\right)+\left(2p_{1\{2,3\}}+2p_{1\{2\}}p_{1\{3\}}-\left(p_{1\{2,3\}}\right)^{2}\right)\right.$$

$$\left.\left\{\alpha\left(p_{2\phi}+p_{2\{3\}}\right)+\beta\right\}p_{3(4)}+\alpha^{3}\left(2p_{1\phi}p_{1\{2\}}+\left(p_{1\{2\}}\right)^{2}\right)p_{2\{3\}}p_{3\{4\}}\right]z^{00010}.$$

The system survivability is equal to the coefficient corresponding to the single term of the polynomial:

$$S_1=\left\{\alpha^{3}\left(2p_{1\{2,3\}}+2p_{1\{2\}}p_{1\{3\}}-\left(p_{1\{2,3\}}\right)^{2}\right)\left(p_{2\{5\}}+p_{2\{3,5\}}\right)p_{3\{4\}}+\alpha^{3}\left(2p_{1\phi}p_{1\{2\}}+\left(p_{1\{2\}}\right)^{2}\right)\right.$$

$$\left.p_{2\{3,5\}}p_{3\{4\}}\right\}$$

$$=\alpha^{3}\left(2p_{1\{2,3\}}+2p_{1\{2\}}p_{1\{3\}}-\left(p_{1\{2,3\}}\right)^{2}\right)\left(p_{2\{5\}}+p_{2\{3,5\}}\right)p_{3\{4\}}+\alpha^{3}\left(2\left(1-p_{1\{2\}}-p_{1\{3\}}-p_{1\{2,3\}}\right)\right.$$

$$p_{1\{2\}}+\left(p_{1\{2\}}\right)^2\Big)p_{2\{3,5\}}p_{3\{4\}}\Big\}$$

$$=2\alpha^3 p_{1\{2,3\}}p_{3\{4\}}\left(p_{2\{5\}}+p_{2\{3,5\}}-p_{1\{2\}}p_{2\{3,5\}}\right)-\alpha^3\left(p_{1\{2,3\}}\right)^2 p_{3\{4\}}\left(p_{2\{5\}}+p_{2\{3,5\}}\right)$$

$$+2\alpha^3 p_{1\{2\}}p_{3\{4\}}\left(p_{1\{3\}}p_{2\{5\}}+p_{2\{3,5\}}\right)-\alpha^3 p_{1\{2\}}p_{2\{3,5\}}p_{3\{4\}}\left(p_{1\{2\}}+2p_{1\{2,3\}}\right)$$

$$=\alpha^3\Big\{2p_{1\{2,3\}}p_{3\{4\}}\left(p_{2\{5\}}+p_{2\{3,5\}}-p_{1\{2\}}p_{2\{3,5\}}\right)-\left(p_{1\{2,3\}}\right)^2 p_{3\{4\}}\left(p_{2\{5\}}+p_{2\{3,5\}}\right)$$

$$+2p_{1\{2\}}p_{3\{4\}}\left(p_{1\{3\}}p_{2\{5\}}+p_{2\{3,5\}}\right)-p_{1\{2\}}p_{2\{3,5\}}p_{3\{4\}}\left(p_{1\{2\}}+2p_{1\{2,3\}}\right)\Big\}.$$

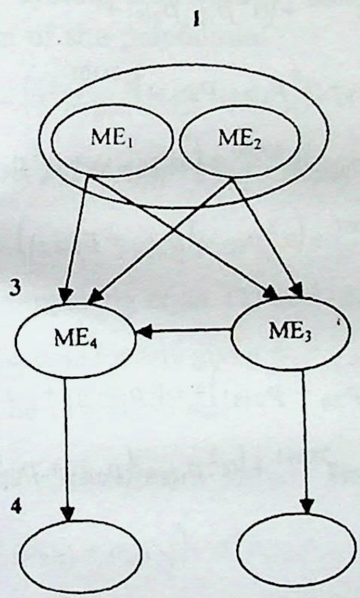


Fig.1.

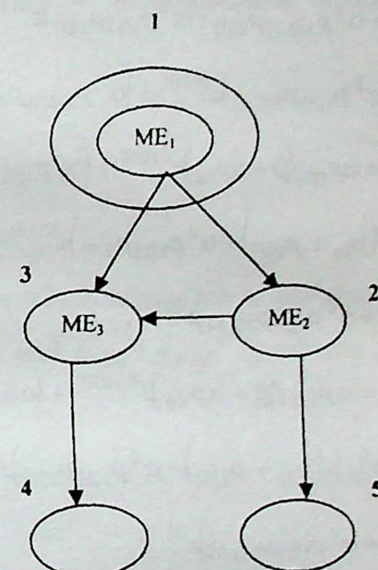


Fig.2.

AMTNs For The Survivability Analysis.

Case-II. Let us consider *AMTN* with $N=5$ and $M=2$ presented in *Fig*. 2 individual *MEs* located at node C_1, C_2 and C_3 are, here the number of *MEs* is equal to the number of non-leaf positions.

$$\widetilde{U}_1(z)=\theta(u_1(z))=\left(\beta+\alpha p_{1\phi}\right)z^{00000}+\alpha p_{1\{2\}}z^{01000}+\alpha p_{1\{3\}}z^{00100}+\alpha p_{1\{2,3\}}z^{01100},$$

$$\widetilde{U}_2(z)=\theta(u_2(z))=\left(\beta+\alpha p_{2\phi}\right)z^{00000}+\alpha p_{2\{3\}}z^{00100}+\alpha p_{2\{5\}}z^{00001}+\alpha p_{2\{3,5\}}z^{00101},$$

$$\widetilde{U}_3(z)=\theta(u_3(z))=\left(\beta+\alpha p_{3\phi}\right)z^{00000}+\alpha p_{3\{4\}}z^{00010}.$$

Following the consecutive procedure, we obtain :

$$\widetilde{U}_1(z)=\hat{U}_1(z),$$

$$\varphi\left(\hat{U}_1(z)\right) = \alpha p_{1\{2\}} z^{01000} + \alpha p_{1\{3\}} z^{00100} + \alpha p_{1\{2,3\}} z^{01100},$$

$$\hat{U}_2(z) = \Psi\left(\varphi\left(\hat{U}_1(z)\right), \widetilde{U}_2(z)\right)$$

$$= \Psi\left(\alpha p_{1\{2\}} z^{01000} + \alpha p_{1\{3\}} z^{00100}\right) + \alpha p_{1\{2,3\}} z^{01100}, (\beta + \alpha p_{2\phi})z^{00000} + \alpha p_{2\{3\}} z^{00100}$$

$$+ \alpha p_{2\{5\}} z^{00001} + a p_{2\{3,5\}} z^{00101}\Big)$$

$$= \alpha p_{1\{2\}}(\beta + \alpha p_{2\phi})z^{01000} + \left(\alpha p_{1\{3\}}(\beta + \alpha p_{2\phi}) + \alpha^2 p_{1\{3\}} p_{2\{3\}}\right)z^{00100} + \left(\alpha p_{1\{2,3\}}(\beta + \alpha p_{2\phi})\right.$$

$$+ \alpha^2 p_{1\{2,3\}} p_{2\{3\}} + \alpha^2 p_{1\{2\}} p_{2\{3\}}\Big)z^{01100} + \alpha^2 p_{1\{2\}} p_{2\{5\}} z^{01001} + \left(\alpha^2 p_{1\{3\}} p_{2\{5\}} + \right.$$

$$\alpha^2 p_{1\{3\}} p_{2\{3,5\}}\Big)z^{00101} + \left(\alpha^2 p_{1\{2,3\}} p_{2\{5\}} + \alpha^2 p_{1\{2\}} p_{2\{3,5\}} + \alpha^2 p_{1\{2,3\}} p_{2\{3,5\}}\right)z^{01101}.$$

$$= \alpha p_{1\{2\}}(\beta + \alpha p_{2\phi})z^{01000} + \left(\alpha\beta p_{1\{3\}} + \alpha^2 p_{1\{3\}}(p_{2\phi} + p_{2\{3\}})\right)z^{00100} + \left(\alpha\beta p_{1\{2,3\}} + \alpha^2 p_{1\{2,3\}}\right.$$

$$(p_{2\phi} + p_{2\{3\}}) + \alpha^2 p_{1\{2\}} p_{2\{3\}}\Big)z^{01101} + \alpha^2 p_{1\{2\}} p_{2\{5\}} z^{01001} + \left(\alpha^2 p_{1\{2,3\}}(p_{2\{5\}} + p_{2\{3,5\}})\right.$$

$$+ \alpha^2 p_{1\{2\}} p_{2\{3,5\}}\Big)z^{01101}.$$

$$= \alpha p_{1\{2\}}(\beta + \alpha p_{2\phi})z^{01000} + \left(\alpha\beta p_{1\{3\}} + \alpha^2 p_{1\{3\}}(p_{2\phi} + p_{2\{3\}}) + \alpha\beta p_{1\{2,3\}} + \right.$$

$$\alpha^2 p_{1\{2,3\}}(p_{2\phi} + p_{2\{3\}}) + \alpha^2 p_{1\{2\}} p_{2\{3\}}\Big)z^{00100} + \alpha^2 p_{1\{2\}} p_{2\{5\}} z^{00001} + \left(\alpha^2 p_{1\{2,3\}}(p_{2\{5\}} + p_{2\{3,5\}})\right.$$

$$+ \alpha^2 p_{1\{2\}} p_{2\{3,5\}}\Big)z^{00101}.$$

$$\varphi\left(\hat{U}_2(z)\right) = \left(\alpha\beta p_{1\{3\}} + \alpha^2 p_{1\{3\}}(p_{2\phi} + p_{2\{3\}}) + \alpha\beta p_{1\{2,3\}} + \alpha^2 p_{1\{2,3\}}(p_{2\phi} + p_{2\{3\}}) + \alpha^2 p_{1\{2\}} p_{2\{3\}}\right)$$

$$z^{00100} + \alpha^2 p_{1\{2\}} p_{2\{5\}} z^{00001} + \left(\alpha^2 p_{1\{2,3\}}\left(p_{2\{5\}} + p_{2\{3,5\}} + \alpha^2 p_{1\{2\}} p_{2\{3,5\}}\right)\right)z^{00101}.$$

$$\hat{U}_3(z) = \Psi\left(\varphi\left(\hat{U}_2(z)\right), \widetilde{U}_3(z)\right),$$

$$= \Psi\left(\alpha\beta p_{1\{3\}} + \alpha^2 p_{1\{3\}}(p_{2\phi} + p_{2\{3\}}) + \alpha\beta p_{1\{2,3\}} + \alpha^2 p_{1\{2,3\}}(p_{2\phi} + p_{2\{3\}}) + \alpha^2 p_{1\{2\}} p_{2\{3\}}\right)$$

$$z^{00100} + \alpha^2 p_{1\{2\}} p_{2\{5\}} z^{00001} + \left(\alpha^2 p_{1\{2,3\}}(p_{2\{5\}} + p_{2\{3,5\}}) + \alpha^2 p_{1\{2\}} p_{2\{3,5\}}\right)z^{00101},$$

$$(\beta + \alpha p_{3\phi})z^{00000} + \alpha p_{3[4]} z^{00010}\Big)$$

$$= \left(\alpha\beta p_{1\{3\}} + \alpha^2 p_{1\{3\}}(p_{2\phi} + p_{2\{3\}}) + \alpha\beta p_{1\{2,3\}} + \alpha^2 p_{1\{2,3\}}(p_{2\phi} + p_{2\{3\}}) + \alpha^2 p_{1\{2\}} p_{2\{3\}}\right)$$

$$(\beta + \alpha p_{3\phi})z^{00100} + \left(\alpha^2 p_{1\{2,3\}}(p_{2\{5\}} + p_{2\{3,5\}}) + \alpha^2 p_{1\{2\}} p_{2\{3,5\}}\right)(\beta + \alpha p_{3\phi})z^{00101} +$$

$$\left(\alpha^3 p_{1\{2,3\}}\left(p_{2\{5\}} + p_{2\{3,5\}}\right) + \alpha^3 p_{1\{2\}} p_{2\{3,5\}}\right) p_{3\{4\}} z^{00011} + \left(\alpha^2 \beta p_{1\{3\}} + \alpha^3 p_{1\{3\}}\left(p_{2\phi} + p_{2\{3\}}\right)\right.$$

$$\left. + \alpha^2 \beta p_{1\{2,3\}}\left(p_{2\phi} + p_{2(3)}\right) + \alpha^3 p_{1\{2\}} p_{2\{3\}}\right) p_{3\{4\}} z^{00110}.$$

$$\varphi\left(\hat{U}_3(z)\right) = \left(\alpha^2 p_{1\{2,3\}}\left(p_{2\{5\}} + p_{2\{3,5\}}\right) + \alpha^2 p_{1\{2\}} p_{2\{3,5\}}\right)\left(\beta + \alpha p_{3\phi}\right) z^{00001} + \left(\alpha^3 p_{1\{2,3\}}\right.$$

$$\left.\left(p_{2\{5\}} + p_{2\{3,5\}}\right) + \alpha^3 p_{1\{2\}} p_{2\{3,5\}}\right) p_{3\{4\}} z^{00011} + \left(\alpha^2 \beta p_{1\{3\}} + \alpha^3 p_{1\{3\}}\left(p_{2\phi} + p_{2\{3\}}\right)\right.$$

$$\left. + \alpha^2 \beta p_{1\{2,3\}} + \alpha^3 p_{1\{2,3\}}\left(p_{2\phi} + p_{2\{3\}}\right) + \alpha^3 p_{1\{2\}} p_{2\{3\}}\right) p_{3\{4\}} z^{00010}.$$

The system survivability is equal to the coefficient corresponding to the single term of the polynomial

$$S_n = \left(\alpha^3 p_{1\{2,3\}}\left(p_{2\{5\}} + p_{2\{3,5\}}\right) + \alpha^3 p_{1\{2\}} p_{2\{3,5\}}\right) p_{3\{4\}} \tag{2}$$

Since $p_{2\phi} + p_{2\{5\}} + p_{2\{3\}} + p_{2\{3,5\}} = 1$

$$= \left(\alpha^3 p_{1\{2,3\}}\left(1 - p_{2\phi} - p_{2\{3\}}\right) + \alpha^3 p_{1\{2\}} p_{2\{3,5\}}\right) p_{3\{4\}}. \tag{3}$$

By comparing eqns. (1) and (2), one can decide which allocation of the elements is preferable for any given $\alpha, p_{1\{2,3\}}, p_{3\{4\}}, p_{1\{2\}}, p_{1\{3\}}$ and $p_{2\{5\}} + p_{2\{3,5\}}$. Condition $S_1 \geq S_n$ can be rewritten as,

$$2\alpha^3 p_{1\{2,3\}} p_{3\{4\}}\left(p_{2\{5\}} + p_{2\{3,5\}} - p_{1\{2\}} p_{2\{3,5\}}\right) - \alpha^3\left(p_{1\{2,3\}}\right)^2 p_{3\{4\}}\left(p_{2\{5\}} + p_{2\{3,5\}}\right) + 2\alpha^3 p_{1\{2\}} p_{3\{4\}}$$

$$\left(p_{1\{3\}} p_{2\{5\}} + p_{2\{3,5\}}\right) - \alpha^3 p_{1\{2\}} p_{2\{3,5\}} p_{3\{4\}}\left(p_{1\{2\}} + 2 p_{1\{2,3\}}\right) \geq \left(\alpha^3 p_{1\{2,3\}}\left(p_{2\{5\}} + p_{2\{3,5\}}\right)\right.$$

$$\left. + \alpha^3 p_{1\{2\}} p_{2\{3,5\}}\right) p_{3\{4\}} \tag{4}$$

Equation (4) present $S_1 \geq S_n$ as function of variables $p_{1\{2,3\}} p_{3\{4\}}$ and $\left(p_{2\{5\}} + p_{2\{3,5\}}\right)$. Solution I provides lower system survivability than solution II, when the values of α located below the curve and the solution II provides lower system survivability than solution I, when the values of α located above the curve.

4. Conclusion. This paper is based on universal generating function technique for solving the survivability of *AMTNs*. The resulting polynomial contains $2^M - 1$ term. Therefore, the suggested method can be applied for *AMTNs* with moderate values of *M* where *M* is the number of leaf nodes. The algorithm which is used in this paper for solving *AMTNs* survivability of *MEs* in which the nodes vulnerability is taken into account. The networks are arranged in such a way that signal transmission from root node to leaf node. The paper suggests a system survivability method for comparison of two networks.

REFERENCES

[1] R.S. Gaur and S. Chaudhary, Availability and maximizing survivability of Cable T.V. transmission system, *Jñānābha*, **34** (2004), 107-112.

[2] R.S. Gaur and S. Chaudhary, Reliability enhancement and fixed transmission times of acyclic transmission multi-state networks (*AMTNs*). *Proceedings of the Ramunujan Mathematical Society.* Agra (2004).

[3] R.S. Gaur and S. Chaudhary, Reliability enhancement and fixed transmission times of acyclic transmission multi-state networks (*ATMNs*), *JMASS*, **2** (2006), 46-56.

[4] F. Hwang and Y. Yao, Multi-state consecutively-connected systems, *IEEE Transaction on Reliability*, **38**(1989), 472-474.

[5] A Kossow and W. Preuss, Reliability of linear consecutively-connected systems with multi-state components, *IEEE Transaction on Reliability,* **44** (1995), 518-522.

[6] G. Levitin, Optimal allocation of multi-state retransmitters in acyclic transmission networks, *Reliability Engineering and System Safety,* **75** (2002), 73-82.

[7] G. Levitin, Reliability evaluation for acyclic consecutively-connected networks with multi-state elements, *Reliability Engineering and System Safety,* **73** (2001), 137-143.

[8] A. Lisnianski and G. Levitin, Multi-State System Reliability: Assessment, Optimization and Applications, Vol. 6: Series on Quality,Reliability & Engineering, Word Scientific, New Jersey, (2003).

[9] J. Malinowski and W. Preuss, Reliability evaluation for tree-structured systems with multi-state components, *Microelectron Reliab,* **36** (1996), 9-17.

[10] M. Zuo and M. Liang, Reliability of multi-state consecutively-connected systems, *Reliability Engineering and system Safety,* **44** (1994), 173-176.

Jñānābha, Vol. 38, 2008

THE INTEGRATION OF CERTAIN PRODUCTS INVOLVING H-FUNCTION WITH GENERAL POLYNOMIALS AND INTEGRAL FUNCTION OF TWO COMPLEX VARIABLES

By

V.G. Gupta
Department of Mathematics
University of Rajasthan, Jaipur-302004,Rajasthan
and
Nawal Kishor Jangid
Department of Mathematics
Global Institute of Technology, Jaipur-302022, Rajasthan

(Received : August 14, 2008*)*

ABSTRACT

In this paper, we wstablish a new integral involving general polynomials of Srivastava (1985), Laguere polynomials and integral function of two complex variables of order ρ, given in Dzrbasjan (1957), based on the properties of Fox's H-function. This Integral is unified in nature and acts as a key formula from which we can derive as its specials cases, integrals pertaining to a large number of simpler special functions and polynomials. For example, we derive a few special cases of our integral and main theorem which are also new and of interest by themselves. The results established here are basis in nature and are likely to be of useful applications in several field's notably mathematical physics, statistical mechanics and probability theory.

2000 Mathematics Subject Classification : Primary 33C70; Secondary 33C60
Keywords : General polynomials, Laguerre polynomial and Fox's H-function.

1. Introduction. Srivastava ([11], P. 185, eq.(7)) has defined and introduced the general polynomials

$$S_{U_1,\ldots,U_r}^{V_1,\ldots,V_r}(x_1,\ldots,x_R)=\sum_{k_1=0}^{[U_1/V_1]}\cdots\sum_{k_R=0}^{[U_r/V_r]}\frac{(-U_1)_{V_1k_1}}{k_1!}\cdots\frac{(-U_R)_{V_Rk_R}}{k_R!}\times A[U_1,k_1;\ldots;U_R,k_R]x_1^{k_1}\ldots x_R^{k_R} \quad (1.1)$$

where $\upsilon_1,\ldots,\upsilon_r$ are arbitrary positive integers and the coefficients $A[U_1,k_1;\ldots;U_r,k_r]$ are arbitrary constants real or complex.

The Fox's H-function is defined in terms of a Mellin-Barnes type integral as

$$H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{matrix}\right.\right]=\frac{1}{2\pi\omega}\int_L \phi(s)z^s ds \quad \ldots(1.2)$$

Where $\phi(s) = \dfrac{\prod_{j=1}^{m} \Gamma(b_j - \beta_j s) \prod_{j=1}^{n} \Gamma(1 - a_j + \alpha_j s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + \beta_j s) \prod_{j=n+1}^{p} \Gamma(a_j - \alpha_j s)}$ ...(1.3)

$\omega = \sqrt{-1}, z \neq 0$. An empty product is interpreted as unity. Also m,n,p and q are integers satisfying $1 \le m \le q$, $0 \le n \le p$, $(j = 1,\ldots,p), \beta_j (j = 1,\ldots,q)$ are positive numbers and $a_j (j = 1,\ldots,p), b_j (j = 1,\ldots,q)$ are complex numbers. L is the suitable contour of Barnes type such that the poles of $\Gamma(b_j - \beta_j s)(j = 1,\ldots,m)$ lie to the right and those of $\Gamma(1 - a_j + \alpha_j s)(j = 1,\ldots,n)$ lie to the left of L. the Integral converges if

$$\sum_{j=1}^{p} \alpha_j - \sum_{j=1}^{q} \beta_j \equiv A \le 0, \sum_{j=1}^{n} \alpha_j - \sum_{j=n+1}^{p} \alpha_j + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q} \beta_j \equiv B > 0$$

and $|\arg(z)| B\pi/2$. These assumptions for the H-function will be adhered to throughout the paper. $H_{p,q}^{m,n}\left[z \left| \begin{matrix} \{(a_p, \alpha_p)\} \\ \{(b_q, \beta_q)\} \end{matrix} \right. \right]$, where $\{(a_p, \alpha_p)\}$ stands for $(a_1, \alpha_1), \ldots, (a_p, \alpha_p)$ and $\{(b_q, \beta_q)\}$ stands for $(b_1, \beta_1), \ldots, (b_q, \beta_q)$.

The symbol $\Delta(m,n)$ represents for the set of m parameters:

$$\frac{n}{m}, \frac{n+1}{m}, \ldots, \frac{n+m-1}{m}.$$

Let $F(z_1, z_2) = \sum_{n_1, n_2 = 0}^{\infty} \dfrac{a_{n_1,n_2}}{n_1! n_2!} z_1^{n_1} z_2^{n_2}$ ...(1.4)

be an integral function of complex variables z_1 and z_2. Denote $M_F(r) = \max_{|z_1|+|z_2|=r} |F(z_1, z_2)|$ the maximum modulus of $F(z_1, z_2)$.

Dzrabasjan ([4]. P. 257) has given the following definition of order:

The integral function $F(z_1, z_2)$ is said to be of order ρ if

$$\lim_{r \to \infty} Sup. \frac{\log \log M_F(r)}{\log r} = \rho (0 \le \rho < \infty) \qquad \text{...(1.5)}$$

2. First Integral. In this section, we evaluate the following integral which will be required in our Investigation :

$$\int_0^\infty x^y e^{-x} L_k^{(\sigma)}(x) S_{U_1,\ldots,U_R}^{V_1,\ldots,V_R}\left[y_1 x^\delta,\ldots,y_R x^\delta\right] H_{p,q}^{m,n}\left[zx^\delta \left| \begin{matrix} \{(a_p,\alpha_p)\} \\ \{(b_q,\beta_q)\} \end{matrix} \right.\right] dx$$

$$= (-1)^k (2\pi)^{\frac{1}{2}(1-\delta)} \frac{\delta^{\gamma+\theta+k+1/2}}{k!} L(y_1,\ldots,y_R)$$

$$\times H_{p,2\delta,q+\delta}^{m,n+2\delta}\left[z\delta^\delta \left| \begin{matrix} (\Delta(\delta,-\gamma-\theta),1)(\Delta(\delta,\sigma-\gamma-\theta),-1)\{(a_p,\alpha_p)\} \\ \{(b_q,\beta_q)\},(\Delta(\delta,\sigma-\gamma-\theta+k),1) \end{matrix} \right.\right] \qquad \ldots(2.1)$$

where δ is a positive integer, $A \le 0, B > 0, |\arg z| < B\pi/2$ and

$\mathrm{Re}(\gamma+\theta+1+\delta(b_j/\beta_j)) > -1 (j=1,\ldots,m), \theta = \delta\sum_{j=1}^{R} k_j$ and

$$L(y_1,\ldots,y_R) = \sum_{k_1=0}^{[U_1/V_1]}\cdots\sum_{k_R=0}^{[U_r/V_r]}\prod_{j=1}^{R}\left[\frac{(-U_j)_{V_j k_j}\, y_j^{k_j}}{k_j!}\right] A[U_1,k_1;\ldots;U_R,k_R].$$

Proof. To prove (2.1) we first express the general polynomials by (1.1), the H-function by (1.2), then change the order of integration and summation which is justifiable due to the convergence conditions mentioned with the integral and evaluate the inner integral with the help of ([7], p 76), now using the formula

$$\frac{\Gamma(1-\alpha-n)}{\Gamma(1-\alpha)} = (-1)^n \frac{\Gamma(\alpha)}{\Gamma(\alpha+n)}$$

and the Gauss integral multiplication formula ([5], p.4) we arrive at the right hand side of (2.1) after a little simplification.

3. Main Theorem. Let $|\zeta_l| \ne 0, |\arg\zeta_l| < \frac{\pi}{2\rho} (l=1,2)$: and let

$F(z_1,z_2) = \sum_{n_1,n_2=0}^{\infty} \frac{a_{n_1,n_2}}{n_1!n_2!} z_1^{n_1} z_2^{n_2}$ be an integral function of two complex variables z_1 and z_2 of order $\rho (0<\rho<\infty)$ then for arg ζ_1 = arg ζ_2, we have

$$P_{n_1,n_2}(\zeta_1,\zeta_2) = \int_0^\infty\int_0^\infty (t_1\zeta_1+t_2\zeta_2)^{\gamma\rho-1} e^{-(t_1\zeta_1+t_2\zeta_2)^\rho} H_{p,q}^{m,n}\left[z(t_1\zeta_1+t_2\zeta_2)^{\delta\rho} \left| \begin{matrix} \{(a_p,\alpha_p)\} \\ \{(b_q,\beta_q)\} \end{matrix} \right.\right]$$

$$\times L_k^{(\sigma)}\left[(t_1\zeta_1+t_2\zeta_2)^\rho\right] S_{U_1,\ldots,U_R}^{V_1,\ldots,V_R}\left[y_1(t_1\zeta_1+t_2\zeta_2)^{\delta\rho},\ldots,y_R(t_1\zeta_1+t_2\zeta_2)^{\delta\rho}\right]$$

$$\times F(t_1,t_2)\,dt_1\,dt_2$$

$$= (-1)^k \frac{(2\pi)^{1/2(1-\delta)}\delta^{\gamma+\theta+k-1/2}}{\rho k!} \sum_{n_1,n_2=0}^{\infty} \frac{a_{n_1 n_2}}{(n_1+n_2+1)!} \frac{\delta^{\frac{n_1+n_2+1}{\rho}}}{\zeta_1^{n_1+1}\zeta_2^{n_2+1}} L[y_1,\ldots,y_R]$$

$$\times H^{m,n+2\delta}_{p,2\delta,q+\delta}\left[z\delta^{\delta} \left| \begin{matrix} \left(\Delta\left(\delta,-\gamma-\theta-\frac{n_1+n_2+1}{\rho}+1\right),1\right)\left(\Delta\left(\delta,\sigma-\gamma-\theta-\frac{n_1+n_2+1}{\rho}+1\right),1\right)\{(a_p,\alpha_p)\} \\ \{(b_q,\beta_q)\},\left(\Delta\left(b,\sigma-\gamma-\theta-\frac{n_1+n_2+1}{\rho}+k+1\right),1\right) \end{matrix} \right. \right]$$

...(3.1)

Provided (i) δ is a +ive integer

(ii) $A \le 0, B < 0, |\arg(z)| < B\pi/2$

(iii) $\mathrm{Re}\left[\gamma+\theta+\frac{n_1+n_2+1}{\rho}+\delta(b_j/\beta_j)\right] > -1, (j=1,\ldots,m)$

(iv) The series in (3.1) is uniformly and absolutely convergent in a suitably choosen domain.

Proof of Theorem. Let us first take $\mathrm{Re}(a) > 0$ and consider the integral

$$I_{n_1,n_2}(a) = \int_0^\infty \int_0^\infty (x_1+x_2)^{\gamma\rho-1} e^{-a(x_1+x_2)^\rho} L_k^{(\sigma)}\left[a(x_1+x_2)^\rho\right]$$

$$S^{V_1,\ldots,V_R}_{U_1,\ldots,U_R}\left[y_1 a^\delta (x_1+x_2)^{\delta\rho},\ldots,y_R a^\delta (x_1+x_2)^{\delta\rho}\right]$$

$$\times H^{m,n}_{p,q}\left[za^\delta (x_1+x_2)^{\delta\rho} \left| \begin{matrix} (a_p,\alpha_p) \\ (b_q,\beta_q) \end{matrix} \right. \right] x_1^{n_1} x_2^{n_2}\, dx_1 dx_2$$

where δ, n_1 and n_2 are positive integers. Changing the variables

$x_1 = t(1-u), x_2 = tu, 0 \le u \le 1, 0 \le t < \infty$

we have

$$I_{n_1,n_2}(a) = \int_0^\infty \int_0^\infty t^{n_1+n_2+\gamma\rho} e^{-at^\rho} L_k^{(\sigma)}(at^\rho) S^{V_1,\ldots,V_R}_{U_1,\ldots,U_R}\left[y_1(at^\rho)^\delta,\ldots,y_R(at^\rho)^\delta\right]$$

$$\times H^{m,n}_{p,q}\left[z(at^\rho)^\delta \left| \begin{matrix} (a_p,\alpha_p) \\ (b_q,\beta_q) \end{matrix} \right. \right] (1-u)^{n_1} u^{n_2} \frac{\partial(x_1,x_2)}{\partial(t,u)}\, dt du$$

$$= \int_0^\infty \int_0^1 t^{n_1+n_2+\gamma\rho} e^{-at^\rho} L_k^{(\sigma)}(at^\rho) S^{V_1,\ldots,V_R}_{U_1,\ldots,U_R}\left[y_1(at^\rho)^\delta,\ldots,y_R(at^\rho)^\delta\right]$$

$$\times H_{p,q}^{m,n}\left[z\left(at^{\rho}\right)^{\delta}\left|\begin{matrix}(a_p,\alpha_p)\\(b_q,\beta_q)\end{matrix}\right.\right](1-u)^{n_1}u^{n_2}\,dt\,du\,.$$

Evaluating u-integral with the help of the Eulerian integral of the first kind (Copson, 1961, P. 212), we obtain

$$I_{n_1,n_2}(a)=\frac{n_1!n_2!}{(n_1+n_2+1)!}\int_0^{\infty}t^{n_1+n_2+\gamma\rho}e^{-at^{\rho}}L_k^{(\sigma)}\left(at^{\rho}\right)S_{U_1,\dots,U_R}^{V_1,\dots,V_R}\left[y_1\left(at^{\rho}\right)^{\delta},\dots,y_R\left(at^{\rho}\right)^{\delta}\right]$$

$$\times H_{p,q}^{m,n}\left[z\left(at^{\rho}\right)^{\delta}\left|\begin{matrix}\{(a_p,\alpha_p)\}\\\{(b_q,\beta_q)\}\end{matrix}\right.\right]dt$$

$$=\frac{n_1!n_2!}{\rho(n_1+n_2+1)!}a^{-\gamma-\frac{n_1+n_2+1}{\rho}}\int_0^{\infty}x^{\gamma+\frac{n_1+n_2+1}{\rho}-1}e^{-x}L_k^{(\sigma)}(x)$$

$$\times S_{U_1,\dots,U_R}^{V_1,\dots,V_R}\left[y_1x^{\delta},\dots,y_Rx^{\delta}\right]H_{p,q}^{m,n}\left[zx^{\delta}\left|\begin{matrix}\{(a_p,\alpha_p)\}\\\{(b_q,\beta_q)\}\end{matrix}\right.\right]dx\,. \qquad \dots(3.2)$$

Now evaluating x-integral with the help of (2.1) we have

$$I_{n_1,n_2}(a)=\frac{(-1)^k(2\pi)^{\frac{1}{2}(1-\delta)}}{\rho}L(y_1,\dots,y_R)\frac{n_1!n_2!}{k!(n_1+n_2+1)!}\times\frac{\delta^{\gamma+\theta+\frac{n_1+n_2+1}{\rho}+k-\frac{1}{2}}}{a^{\gamma+\frac{n_1+n_2+1}{\rho}}}$$

$$\times H_{p,2\delta,q+\delta}^{m,n+2\delta}\left[z\delta^{\delta}\left|\begin{matrix}\left(\Delta\left(\delta,-\gamma-\theta-\frac{n_1+n_2+1}{\rho}+1\right);1\right)\left(\Delta\left(\delta,\sigma-\gamma-\theta-\frac{n_1+n_2+1}{\rho}+1\right),1\right),\{(a_p,\alpha_p)\}\\\{(b_q,\beta_q)\},\left(\Delta\left(b,\sigma-\gamma-\theta-\frac{n_1+n_2+1}{\rho}+k+1\right),1\right)\end{matrix}\right.\right]$$

$$\dots(3.3)$$

where $A\le 0, B>0, |\arg z|<B\pi/2$.

$$\mathrm{Re}\left[\gamma+\theta+\frac{n_1+n_2+1}{\rho}+\delta\left(b_j/\beta_j\right)\right]>-1,(j=1,\dots,m)\text{ and}$$

$$L(y_1,\dots,y_R)=\sum_{k_1=0}^{[U_1/V_1]}\cdots\sum_{k_R=0}^{[U_r/V_r]}\prod_{j=1}^{R}\left[\frac{(-U_j)_{V_jk_j}}{k_j!}y_j^{k_j}\right]A[U_1,k_1;\dots;U_R,k_R]$$

Let arg $\zeta_1=\arg\zeta_2=\alpha$ and if we denote $a=e^{i\alpha\rho}$ where

$\mathrm{Re}(a)=\cos(\alpha\rho)>0, |\arg z|<B\pi/2$, then from (3.1), we obtain

$$P_{n_1,n_2}(\zeta_1,\zeta_2)=\int_0^{\infty}\int_0^{\infty}a(t_1\zeta_1+t_2\zeta_2)^{\gamma\rho-1}e^{-a(t_1\zeta_1+t_2\zeta_2)^{r}}L_k^{(\sigma)}\left[(t_1\zeta_1+t_2\zeta_2)^{\rho}\right]$$

$$\times S_{U_1,\ldots,U_R}^{V_1,\ldots,V_R}\left[y_1a^{\delta}(t_1\zeta_1+t_2\zeta_2)^{\delta\rho},\ldots,y_Ra^{\delta}(t_1\zeta_1+t_2\zeta_2)^{\delta\rho}\right]$$

$$\times H_{p,q}^{m,n}\left[za^{\delta}(t_1\zeta_1+t_2\zeta_2)^{\delta\rho}\left|\begin{matrix}(a_p,\alpha_p)\\(b_q,\beta_q)\end{matrix}\right.\right]\left(\sum_{n_1,n_2=0}^{\infty}\frac{a_{n_1,n_2}}{n_1!n_2!}t_1^{n_1}t_1^{n_2}\right)dt_1dt_2$$

$$=e^{i\alpha(\gamma\rho-1)}\sum_{n_1,n_2=0}^{\infty}\frac{a_{n_1,n_2}}{n_1!n_2!}\int_0^{\infty}\int_0^{\infty}(t_1|\zeta_1|+t_2|\zeta_2|)^{\gamma\rho-1}\times e^{-a(t_1|\zeta_1|+t_2|\zeta_2|)^{r}}L_k^{\sigma}\left[a(t_1|\zeta_1|+t_2|\zeta_2|)^{\rho}\right]$$

$$\times S_{U_1,\ldots,U_R}^{V_1,\ldots,V_R}\left[y_1a^{\delta}(t_1|\zeta_1|+t_2|\zeta_2|)^{\delta\rho},\ldots,y_Ra^{\delta}(t_1|\zeta_1|+t_2|\zeta_2|)^{\delta\rho}\right]$$

$$\times H_{p,q}^{m,n}\left[za^{\delta}(t_1|\zeta_1|+t_2|\zeta_2|)^{\delta\rho}\left|\begin{matrix}\{(a_p,\alpha_p)\}\\\{(b_q,\beta_q)\}\end{matrix}\right.\right]t_1^{n_1}t_2^{n_2}dt_1dt_2$$

$$=e^{i\alpha(\gamma\rho-1)}\sum_{n_1,n_2=0}^{\infty}\frac{a_{n_1,n_2}}{n_1!n_2!}\times\frac{1}{|\zeta_1|^{n_1+1}\cdot|\zeta_2|^{n_2+1}}\times\int_0^{\infty}\int_0^{\infty}(x_1+x_2)^{\gamma\rho-1}e^{-a(x_1+x_2)^{r}}L_k^{(\sigma)}\left[a(x_1+x_2)^{\rho}\right]$$

$$\times S_{U_1,\ldots,U_R}^{V_1,\ldots,V_R}\left[y_1a^{\delta}(x_1+x_2)^{\delta\rho},\ldots,y_Ra^{\delta}(x_1+x_2)^{\delta\rho}\right]$$

$$\times H_{p,q}^{m,n}\left[za^{\delta}(x_1+x_2)^{\delta\rho}\left|\begin{matrix}(a_p,\alpha_p)\\(b_q,\beta_q)\end{matrix}\right.\right]x_1^{n_1}x_1^{n_2}dx_1dx_2$$

$$=\frac{(-1)^k(2\pi)^{(1-\delta)/2}}{\rho k!}e^{i\alpha(\gamma\rho-1)}\sum_{n_1,n_2=0}^{\infty}\frac{a_{n_1n_2}}{(n_1+n_2+1)!}\times\frac{\delta^{\gamma+\theta+\frac{n_1+n_2+1}{\rho}+k-1/2}}{a^{\gamma+\frac{n_1+n_2+1}{\rho}}}$$

$$\times\frac{1}{|\zeta_1|^{n_1+1}|\zeta_2|^{n_2+1}}L[y_1,\ldots,y_R]$$

$$\times H_{p,2\delta.q+\delta}^{m,n+2\delta}\left[z\delta^{\delta}\left|\begin{matrix}\left(\Delta\left(\delta,-\gamma-\theta-\frac{n_1+n_2+1}{\rho}+1\right),1\right),\left(\Delta\left(\delta,\sigma-\gamma-\theta-\frac{n_1+n_2+1}{\rho}+1\right),1\right),\{(a_p,\alpha_p)\}\\\{(b_q,\beta_q)\},\left(\Delta\left(\Delta,\sigma-\gamma-\theta-\frac{n_1+n_2+1}{\rho}+k+1\right),1\right)\end{matrix}\right.\right]$$

due to relation (3.2) where

$$L(y_1,\ldots,y_R)=\sum_{k_1=0}^{[U_1/V_1]}\cdots\sum_{k_R=0}^{[U_r/V_r]}\prod_{j=1}^{R}\left[(-U_j)_{\frac{v_jk_j}{k_j!}}y_j^{k_j}\right]A[U_1,k_1;\ldots;U_R,k_R].$$

Thus under conditions $|\zeta_1| \neq 0, \arg\zeta_l = \arg(\zeta_2), |\arg\zeta_1| < \frac{\pi}{2\rho}(l=1,2)$, and an appeal to analytic continuation we have (3.1).

The change of order of integration and summations in (3.3) is justified by the de 1 vallee Poussiu's theorem (Bromwich, 1931, p. 504) under the condition imposed in the theorem.

4. Special Cases. (i) If take R=1 in (2.1) we get integral involving the product of general class of polynomials and Fox's H-function.

$$\int_0^\infty x^\gamma e^{-x} L_k^{(\sigma)}(x) S_{U_1}^{V_1}\left[y_1 x^\delta\right] H_{p,q}^{m,n}\left[zx^\delta \left| \begin{matrix} \{(a_p,\alpha_p)\} \\ \{(b_q,\beta_q)\} \end{matrix} \right.\right] dx$$

$$= (-1)^k (2\pi)^{\frac{1}{2}(1-\delta)} \delta^{\gamma+\delta k_1+k+1/2} \sum_{k_1=0}^{[U_1/V_1]} \frac{(-U_j)_{v_j k_j}}{k_1!} y_1^{k_1} A[U_1,k_1]$$

$$\times H_{p,2\delta,q+\delta}^{m,n+2\delta}\left[z\delta^\delta \left| \begin{matrix} (\Delta(\delta,-\gamma-\delta k_1),1),(\Delta(\delta,\sigma-\gamma-\delta k_1),1),\{(a_p,\alpha_p)\} \\ \{(b_q,\beta_q)\},(\Delta(\delta,\sigma-\gamma-\delta k_1+k),1) \end{matrix} \right.\right] \quad ...(4.1)$$

where δ is a positive integers, $A \leq 0, B > 0, |\arg z| < B\pi/2$ and

$\mathrm{Re}(\gamma + \delta k_1 + 1 + \delta(b_j/B_j)) > -1, j = 1$.

(ii) If we take R=1 in (3.1) our result reduces to

$$P_{n_1,n_2}(\zeta_1,\zeta_2) = \int_0^\infty\int_0^\infty (t_1\zeta_1+t_2\zeta_2)^{\gamma\rho-1} e^{a(t_1\zeta_1+t_2\zeta_2)^\rho} L_k^{(\sigma)}\left[(t_1\zeta_1+t_2\zeta_2)^\rho\right] S_{U_1}^{V_1}\left[y_1(t_1\zeta_1+t_2\zeta_2)^{\delta\rho}\right]$$

$$\times H_{p,q}^{m,n}\left[z(t_1\zeta_1+t_2\zeta_2)^{\delta\rho} \left| \begin{matrix} \{(a_p,\alpha_p)\} \\ \{(b_q,\beta_q)\} \end{matrix} \right.\right] \times F(t_1,t_2)\,dt_1\,dt_2$$

$$= (-1)^k (2\pi)^{\frac{1}{2}(1-\delta)} \delta^{\gamma+\theta+k-1/2} \sum_{n_1,n_2=0}^{\infty} \frac{a_{n_1 n_2}}{(n_1+n_2+1)!} \frac{\delta(n_1+n_2+1)/\rho}{\zeta_1^{n_1+1}\zeta_2^{n_2+1}}$$

$$\times \sum_{k_1=0}^{[U_1/V_1]} \frac{(-U_1)_{V_1 k_1}}{k_1!} y_1^{k_1} A[U_1,k_1]$$

$$\times H_{p,2\delta,q+\delta}^{m,n+2\delta}\left[z\delta^\delta \left| \begin{matrix} \left(\Delta\left(\delta,-\gamma-\delta k_1-\frac{n_1+n_2+1}{\rho}+1\right),1\right),\left(\Delta\left(\delta,\sigma-\gamma-\delta k_1-\frac{n_1+n_2+1}{\rho}+1\right),1\right),\{(a_p,\alpha_p)\} \\ \{(b_q,\beta_q)\},\left(\Delta\left(\delta,\sigma-\gamma-\delta k_1-\frac{n_1+n_2+1}{\rho}+k+1\right),1\right) \end{matrix} \right.\right]$$

...(4.2)

provided that (i) δ is a positive integer

(ii) $A \leq 0, B < 0, |\arg z| < B\pi/2$

(iii) $\text{Re}\left[\gamma + \delta k_1 + \frac{n_1 + n_2 + 1}{\rho} + \delta(b_j/\beta_j)\right] > -1$

(iv) The series in (4.2) is uniformly and absolutely convergent in a suitably choosen domain.

(iii) Letting $U_1 = 0, A_{0,0} = 1$ in the equation (4.1) and (4.2) we get the result obtained by Shah ([9], p715) and Nigam ([7], p.2, eqn (3.1)) respectively as given below.

$$\int_0^\infty x^\gamma e^{-x} L_k^{(\sigma)}(x) H_{p,q}^{m,n}\left[\left|\begin{matrix}\{(a_p, \alpha_p)\} \\ \{(b_q, \beta_q)\}\end{matrix}\right.\right] dx$$

$$= (-1)^k (2\pi)^{\frac{1}{2}(1-\delta)} \delta^{\gamma+k+1/2} H_{p,2\delta,q+\delta}^{m,n+2\delta}\left[z\delta^\delta \left|\begin{matrix}(\Delta(\delta,-\gamma),1), (\Delta(\delta,\sigma-\gamma),1), \{(a_p,\alpha_p)\} \\ \{(b_q,\beta_q)\}, (\Delta(\delta,\sigma-\gamma+k),1)\end{matrix}\right.\right] \quad \text{...(4.3)}$$

where δ is a positive integer, $A \leq 0, B > 0, |\arg z| < B\pi/2$, and $\text{Re}[\gamma + 1 + \delta(b_j/\beta_j)] > -1$

$$P_{n_1,n_2}(\zeta_1,\zeta_2) = \int_0^\infty \int_0^\infty (t_1\zeta_1 + t_2\zeta_2)^{\gamma\rho-1} e^{(t_1\zeta_1+t_2\zeta_2)^\rho}$$

$$\times L_k^{(\sigma)}[(t_1\zeta_1 + t_2\zeta_2)^\rho] H_{p,q}^{m,n}\left[z(t_1\zeta_1 + t_2\zeta_2)^{\delta\rho}\left|\begin{matrix}\{(a_p,\alpha_p)\} \\ \{(b_q,\beta_q)\}\end{matrix}\right.\right] \times F(t_1,t_2)\,dt_1 dt_2$$

$$= \frac{(-1)^k (2\pi)^{\frac{1}{2}(1-\delta)}}{\rho k!} \delta^{\gamma+k-\frac{1}{2}} \sum_{n_1,n_2=0}^\infty \frac{a_{n_1n_2}}{(n_1+n_2+1)!} \frac{\delta^{\frac{n_1+n_2+1}{\rho}}}{\zeta_1^{n_1+1}\zeta_2^{n_2+1}}$$

$$\times H_{p,2\delta,q+\delta}^{m,n+2\delta}\left[z\delta^\delta\left|\begin{matrix}\left(\Delta\left(\delta,-\gamma-\frac{n_1+n_2+1}{\rho}+1\right),1\right), \left(\Delta\left(\delta,\sigma-\gamma-\frac{n_1+n_2+1}{\rho}+1\right),1\right), \{(a_p,\alpha_p)\} \\ \{(b_q,\beta_q)\}, \left(\Delta\left(\delta,\sigma-\gamma-\frac{n_1+n_2+1}{\rho}+k+1\right),1\right)\end{matrix}\right.\right]$$

...(4.4)

Provided

(i) δ is a positive integer.

(ii) $A \leq 0, B > 0, |\arg z| < B\pi/2$

(iii) $\text{Re}\left[\gamma + \frac{n_1+n_2+1}{\rho} + \delta(b_j/\beta_j)\right] > -1, (j = 1,...,m)$

. (iv) The series in (4.4) is uniformly and absolutely convergent in a suitably chosen domain.

(iv) If we make use of the relation. (Sharma; 1965, p. 199)

$$H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1,1),\ldots,(a_p,1)\\(b_1,1),\ldots,(b_q,1)\end{matrix}\right.\right]=G_{p,q}^{m,n}\left[z\left|\begin{matrix}a_1,\ldots,a_p\\b_1,\ldots,b_q\end{matrix}\right.\right]$$

$$\int_0^\infty\int_0^\infty(t_1\zeta_1+t_2\zeta_2)^{\gamma\rho-1}e^{(t_1\zeta_1+t_2\zeta_2)^\gamma}\times L_k^{(\sigma)}\left[(t_1\zeta_1+t_2\zeta_2)^\rho\right]$$

$$\times S_{U_1,\ldots,U_R}^{V_1,\ldots,V_R}\left[y_1(t_1\zeta_1+t_2\zeta_2)^{\delta\rho},\ldots,y_R(t_1\zeta_1+t_2\zeta_2)^{\delta\rho}\right]$$

$$\times G_{p,q}^{m,n}\left[z\left|(t_1\zeta_1+t_2\zeta_2)^{\delta\rho}\right|\begin{matrix}a_1,\ldots,a_\rho\\b_1,\ldots,b_q\end{matrix}\right]F(t_1,t_2)dt_1dt_2$$

$$=(-1)^k\frac{(2\pi)^{1/2(1-\delta)}\delta^{\gamma+\theta+k-1/2}}{\rho k!}\sum_{n_1,n_2=0}^{\infty}\frac{a_{n_1n_2}}{(n_1+n_2+1)!}\frac{\delta^{\frac{n_1+n_2+1}{\rho}}}{\zeta_1^{n_1+1}\zeta_2^{n_2+1}}\;L[y_1,\ldots,y_R]$$

$$\times G_{p,2\delta,q+\delta}^{m,n+2\delta}\left[z\delta^\delta\left|\begin{matrix}\Delta\left(\delta,-\gamma-\theta-\frac{n_1+n_2+1}{\rho}\right),\Delta\left(\delta,\sigma-\gamma-\theta-\frac{n_1+n_2+1}{\rho}+1\right),\{(a_1,\ldots,\alpha_p)\}\\\{(b_1,\beta_q)\},\Delta\left(\delta,\sigma-\gamma-\theta-\frac{n_1+n_2+1}{\rho}+k+1\right)\end{matrix}\right.\right]\tag{4.5}$$

where

$$L[y_1,\ldots,y_R]=\sum_{k_1=0}^{[U_1/V_1]}\cdots\sum_{k_R=0}^{[U_R/V_R]}\prod_{j=1}^{R}\left[\frac{(-U_j)}{k_j!}y_j^{k_j}\right]A[U_1,k_1;\ldots;U_R,k_R]\text{ and }\theta=\delta\sum_{j=1}^{R}k_j\tag{4.6}$$

(v) If we take R=1 in (4.5) we get the integral involving the product of general class of polynomials and G-function

$$\int_0^\infty\int_0^\infty(t_1\zeta_1+t_2\zeta_2)^{\gamma\rho-1}e^{-(t_1\zeta_1+t_2\zeta_2)^\gamma}\times L_k^{(\sigma)}\left[(t_1\zeta_1+t_2\zeta_2)^\rho\right]\times$$

$$S_{U_1}^{V_1}\left[y_1(t_1\zeta_1+t_2\zeta_2)^{\delta\rho}\right]\times G_{p,q}^{m,n}\left[z\left|(t_1\zeta_1+t_2\zeta_2)^{\delta\rho}\right|\begin{matrix}a_1,\ldots,a_p\\b_1,\ldots,b_q\end{matrix}\right]F(t_1,t_2)dt_1dt_2$$

$$=(-1)^k(2\pi)^{\frac{1}{2}(1-\delta)}\delta^{\gamma+\delta k_1+k+1/2}\sum_{n_1,n_2=0}^{\infty}\frac{a_{n_1n_2}}{(n_1+n_2+1)!}\frac{\delta^{\frac{n_1+n_2+1}{\rho}}}{\zeta_1^{n_1+1}\zeta_2^{n_2+1}}\sum_{k_1=0}^{[U_1/V_1]}\frac{(-U_1)_{V_1k_1}}{k_1!}y_1^{k_1}A[U_1,k_1]$$

$$\times G^{m,n+2\delta}_{p,2\delta,q+\delta}\left[z\delta^{\delta}\left|\begin{matrix}\Delta\left(\delta,-\gamma-\delta k_1-\frac{n_1+n_2+1}{\rho}+1\right),\Delta\left(\delta,\sigma-\gamma-\delta k_1-\frac{n_1+n_2+1}{\rho}+1\right),\{(a_1,\ldots,\alpha_p)\}\\ \{(b_q,\beta_q)\},\Delta\left(\delta,\sigma-\gamma-\delta k_1,-\frac{n_1+n_2+1}{\rho}+k+1\right)\end{matrix}\right.\right]$$

...(4.7)

(vi) Letting $U=0, A_{0,0}=1$ in equation (4.7) we get the result obtained by Nigam ([7], P. 5, eq.(4.1))

$$\int_0^\infty\int_0^\infty (t_1\zeta_1+t_2\zeta_2)^{\gamma\rho-1}e^{-(t_1\zeta_1+t_2\zeta_2)^\rho}\times L_k^{(\sigma)}\left[(t_1\zeta_1+t_2\zeta_2)^\rho\right]\times$$

$$G^{m,n}_{p,q}\left[z|(t_1\zeta_1+t_2\zeta_2)^{\delta\rho}\left|\begin{matrix}a_1,\ldots,a_\rho\\ b_1,\ldots,b_q\end{matrix}\right.\right]F(t_1,t_2)dt_1dt_2$$

$$\frac{(-1)^k(2\pi)^{\frac{1}{2}(1-\delta)}\delta^{\gamma+k-1/2}}{\rho k!}\sum_{n_1,n_2=0}^{\infty}\frac{a_{n_1n_2}}{(n_1+n_2+1)!}\frac{\delta^{\frac{n_1+n_2+1}{\rho}}}{\zeta_1^{n_1+1}\zeta_2^{n_2+1}}$$

$$G^{m,n+2\delta}_{p,2\delta,q+\delta}\left[z\delta^{\delta}\left|\begin{matrix}\Delta\left(\delta,-\gamma-\frac{n_1+n_2+1}{\rho}+1\right),\Delta\left(\delta,\sigma-\gamma-\frac{n_1+n_2+1}{\rho}+1\right),\{(a_1,\ldots,\alpha_p)\}\\ \{(b_q,\beta_q)\},\Delta\left(\delta,\sigma-\gamma-\frac{n_1+n_2+1}{\rho}+k+1\right)\end{matrix}\right.\right].$$

REFERENCES

[1] T.J.I.A. Bromwich, *An Introduction to the Theory of Infinite Series*, London (1931).

[2] B.L.J., Baraaksma, Asymptotic expansions and analytic continuations for a class of Barness-integrals, *Compositio Math.* **15** (1963), 339-341.

[3] E.T. Copson, *Theory of Functions of Complex Variables*, Oxford, (1961).

[4] M.M. Dzrbasjan, *Mathematich Cskii Sbornik*, **41** (1957) (83), 257 (Russian).

[5] A. Erdélyi *et.al.*, *Higher Transcendental Functions*. Mc-Graw Hill Book Company, Inc. New york, (1953).

[6] C. Fox, The *G* and *H*-function as symmetrical Fourier Kernal. *Trans. Amer, Math., Soc.* **98** (1961), 395-429.

[7] A.M. Mathai and R.K. Saxena, Generalized Hypergeometric Functions with Applications in Statistics and Physical Sciences, Springer-verlag, New York, 1970.

[8] H.N. Nigam, Integral involving Fox's *H*-function and integral function of two complex variables, **64** (1972), 1-5.

[9] M. Shah, Some results on the *H*-function involving generalized Laguerre polynomail. Proc, Camb, Phil, Soc. **65** (1969), 713-720.

[10] O.P. Sharma, Some finite and infinite integrals involving *H*-function and Gauss's hypergeometric functions, *Collectanca, Mathematica*, **17** (1965), 197-209.

[11] H.M Srivastava, A multilinear generating function for the Konhauser sets of biorthogonal polynomials suggested by the Laguerre polynomials, *Pacific. J. Math.* **117** (1985), 183-191.

Jñānābha, Vol. 38, 2008

STUDY OF VELOCITY AND DISTRIBUTION OF MAGNETIC FIELD IN LAMINAR STEADY FLOW BETWEEN PARALLEL PLATES

By

Pratap Singh

Department of Mathematics and Statistics, Allahabad, Agricultural Institute-(Deemed University) Allahabad-211007, Uttar Pradesh, India

(Received : October 10, 2008*)*

ABSTRACT

We studied flow of velocity and distribution of magnetic field in laminar steady flow between two parallel plates situded at a distance and having relative velocity between them. For low Hartman number, flow velocity numerically increases rapidly in the middle of the plates, then it numerically increases slowly near the plates. But for large Hartmann number it numerically increases slowly in the middle of the plates and then it numerically increases rapidly near the plates. It decreases the strengh of the magnetic field decreases. For large Hartmann number, the strength of the magnetic field is inversely proportional to the-Hartmann number.

2000 Mathematics Subject Classification: Primerey secondary

Keywords: Magnetic permeability/ coefficient of viscocity/electrical conductivity/ Hartmann number.

1. Introduction. Contribution of laminar steady and unsteady flow has been made by several authors because of its wide application in Engineering, Physical, Medical Science and Oil refining etc. Problem of heat transfer through the annular space when the fluid flow is laminar and there is uniform heating either from out side, from in side or from both was investigated [1]. Numerous theoretical and experimental studies of both laminar and turbulent heat transfer in annuli taking various types of wall temperature distributions have been made ([2],[4]). An analysis on laminar flow and heat transfer in concentric annuli with moving core was obtained [5]. The stability of laminar flow of dusty gas was studied [6]. Contribution on laminar flow an electrically conducted liquid in a homogeneous magnetic field was made [7],[8]. Our problem is to study of velocity and distribution of magnetic field in laminar steady flow between parallel plates situated at a destance haveing relative velocity between them.

2. Formulation of the problem. Let us consider two-dimensional steady laminar flow of an incompressible and electrically conducting fluid of constant viscosity and enstant electrically conductivity between two parallel insulated plates. The two plates are situated at a distance $2L$ and relative velocity $2U$ between

them. The velocity of flow is parallel to the plates, which are in the direction of x-axis. An external magnetic field of constant strength H_0 in the direction of y-axis is also in consideration. Then conditions of flow are:

$$\left.\begin{array}{l} u = Uu^*(y^*), v = 0, w = 0, \\ H_x = H_0 H_x^*(y^*), H_Y = H_0, H_z = 0, \\ P = \rho U^2 p^*(x^*, y^*) \end{array}\right\} \quad \text{...(2.1)}$$

where $x = Lx^*, y = Ly^*, L$ and U are the characteristic length and velocity fot the problem respectively. Using (1) and neglecting stars, the equation of motion

$\nabla.\bar{u} = 0,$

$$\rho\frac{D\bar{u}}{Dt} - \mu_e(\bar{H}.\nabla)\bar{H} = -\nabla\left(p + \frac{\mu_e H^2}{2}\right) + \mu\nabla^2,$$

$$\frac{\bar{H}}{\partial t} + (\bar{u}.\nabla)\bar{H} - (\bar{H}.\nabla)\bar{u} = V_H \nabla^2 \bar{H}$$

takes the form

$$\frac{1}{R_e}\frac{d^2u}{dy^2} + R_H\frac{dH_x}{dy} = \frac{\partial p}{\partial x} \quad \text{...(2.2)}$$

$$R_H H_x \frac{dH_x}{dy} = -\frac{\partial p}{\partial y} \quad \text{...(2.3)}$$

and

$$\frac{du}{dy} + \frac{1}{R_\sigma}\frac{d^2 H_x}{dy^2} = 0 \quad \text{...(2.4)}$$

where $R_e = \dfrac{\rho UL}{\mu}, R_H = \dfrac{\mu_e H_0^2}{\rho v^2}, R_\sigma = \mu_e \sigma UL,$

R_e = Reynolds number,

R_H = Magnetic pressure number,

R_σ = Magnetic Reynolds number,

ρ = density of the fluid,

μ = coefficient of viscosity,

μ_e = Magnetic permeability,

υ = Kinetic coefficient of viscosity,
σ = electrical conductivity.
From (2.2) and (2.3), we get

$$\frac{1}{R_h^2}\frac{d^3u}{dy^3}+\frac{1}{R_\sigma}\frac{d^2H_x}{dy^2}=0 \qquad ...(2.5)$$

where $R_h=\sqrt{R_e R_H R_\sigma}, R_h$ is Hartmann number.

Subtracting (2.4) from (2.5) , we have

$$\frac{d^3u}{dy^3}-R_h^2\frac{du}{dy}=0 \qquad ...(2.6)$$

Consider that the two plates are situated at $y=\pm1$ and there is no pressure gradient in flow field.

Now the boundary conditions are:

$y=1, u=1; y=0, u=0; y=-1, u=-1.$ ...(2.7)

Using condition (2.7) then solution of (2.6) is

$$u=\frac{e^{R_h y}-e^{-R_h y}}{e^{R_h}-e^{-R_h}}. \qquad ...(2.8)$$

Case I. when $R_h\to 0$, we have

$u=y$,

it is a straight line, whose gradient is 1.

Case II. When $R_h\to\infty$, we have

$u=0$ (expect at $y=\pm1$).

Plates are insulated. So the boundary conditions are:

$y=\pm1, H_x=0.$

Integrating (2.4) with respect to t and using above boundary conditions, we get

$$\frac{H_x}{R_\sigma}=\frac{1}{R_h}\left[\frac{\left(e^{R_h}+e^{-R_h}\right)-\left(e^{R_h y}+e^{-R_h y}\right)}{\left(e^{R_h}-e^{-R_h}\right)}\right] \qquad ...(2.9)$$

It is the expression for magnetic field.

3. Results and conclusion. From tabel 1, it is found that for low Hartmann number, the velocity of flow numerically increases rapidly in the middle of the plates, and then it numerically increases slowly near the plates. But for large Hartmann number it numerically increases slowly in the middle of the plates, and then it numerically increases rapidly near the plates. As Hartmann number

Table-1

Velocity of flow is numerically calculated for different Hartmann number.

y/R$_s$	0	2	4	6	8	10
-1.0	-1.0	-1.0	-1.0	-1.0	-1.0	-1.0
-0.9	-0.9	-.8112	-.6700	-.5488	-.4493	-.3679
-0.8	-0.8	-.6550	-.4487	-.3012	-.2019	-.1353
-0.7	-0.7	-.5251	-.3002	-.1653	-.0907	-.0498
-0.6	-0.6	-.4162	-.2003	-.0907	-.0408	-.0183
QW-0.5	-0.5	-.3240	-.1329	-.0497	-.0183	-.0067
-0.4	-0.4	-.2449	-.0871	-.0271	-.0082	-.0025
-0.3	-0.3	-.1755	-.0553	-.0146	-.0037	-.0009
-0.2	-0.2	-.1133	-.0325	-.0075	-.0016	-.0003
-0.1	-0.1	-.0555	-.0151	-.0032	-.0006	-.0001
0	0	0	0	0	0	0
0.1	0.1	.0555	.0151	.0032	.0006	.0001
0.2	0.2	.1133	.0325	.0075	.0016	.0003
0.3	0.3	.1755	.0553	.0146	.0037	.0009
0.4	0.4	.2449	.0871	.0271	.0082	.0025
0.5	0.5	.3240	.1329	.0497	.0183	.0067
0.6	0.6	.4162	.2003	.0907	.0408	.0183
0.7	0.7	.5251	.3002	.1653	.0907	.0498
0.8	0.8	.6550	.4487	.3012	.2019	.1353
0.9	0.9	.8112	.6700	.5488	.4493	.3679
1.0	1.0	1.0	1.0	1.0	1.0	1.0

Figure-1

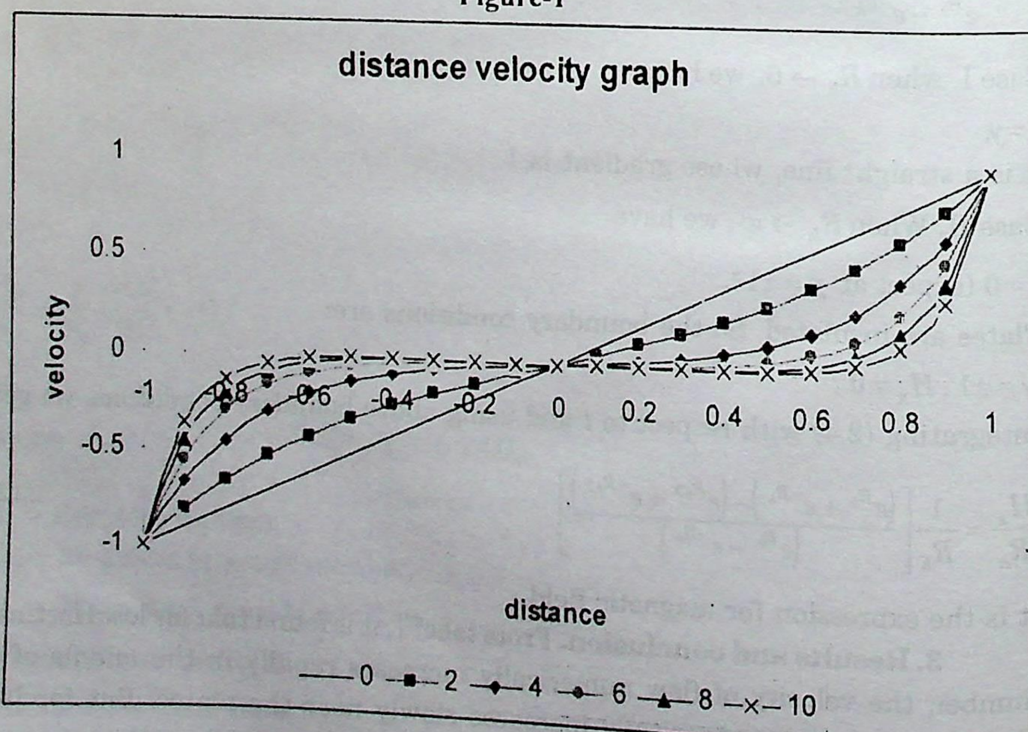
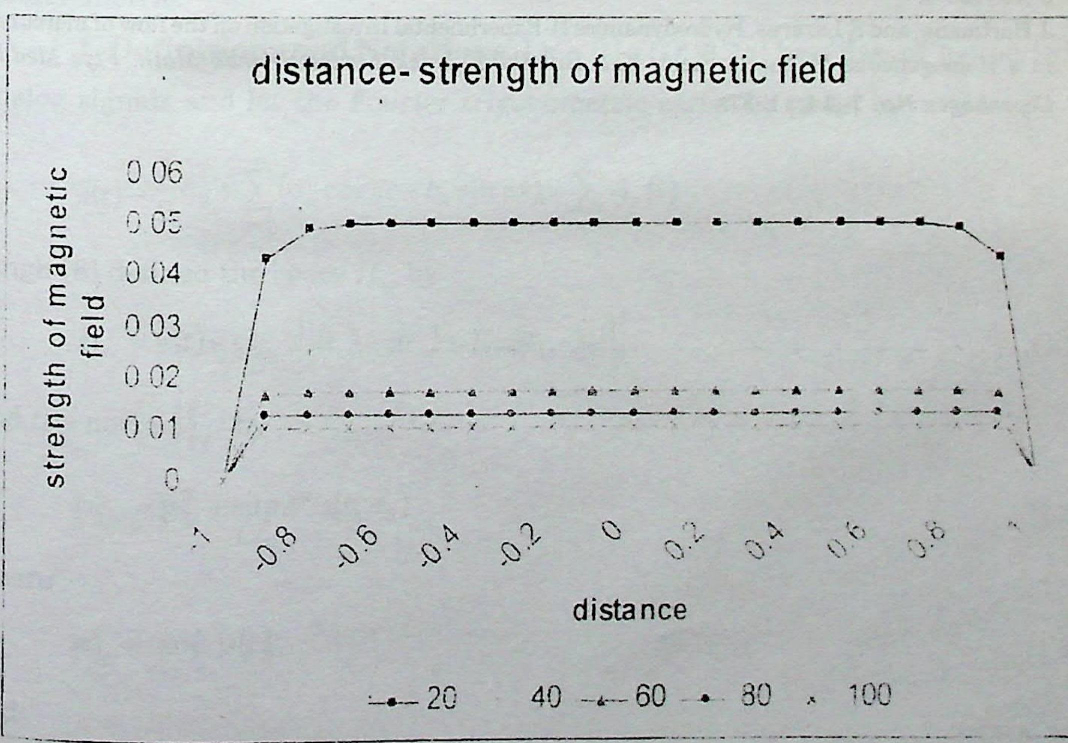


Table – 2

Strength of magnetic field $\frac{H_x}{R_\sigma}$ is numerically calculated for different Hartmann number

y/R_h	20	40	60	80	100
-1.0	0	0	0	0	0
-0.9	.0432	.0254	.0166	.0125	.0100
-0.8	.0491	.0249	.0167	.0125	.0100
-0.7	.0499	.0250	.0167	.0125	.0100
-0.6	.0500	.0250	.0167	.0125	.0100
-0.5	.0500	.0250	.0167	.0125	.0100
-0.4	.0500	.0250	.0167	.0125	.0100
-0.3	.0500	.0250	.0167	.0125	.0100
-0.2	.0500	.0250	.0167	.0125	.0100
-0.1	.0500	.0250	.0167	.0125	.0100
0	.0500	.0250	.0167	.0125	.0100
0.1	.0500	.0250	.0167	.0125	.0100
0.2	.0500	.0250	.0167	.0125	.0100
0.3	.0500	.0250	.0167	.0125	.0100
0.4	.0500	.0250	.0167	.0125	.0100
0.5	.0500	.0250	.0167	.0125	.0100
0.6	.0500	.0250	.0167	.0125	.0100
0.7	.0499	.0250	.0167	.0125	0100
0.8	.0491	.0249	.0167	.0125	.0100
0.9	.0432	.0254	.0166	.0125	.0100
1.0	0	0	0	0	0

Figure-2



increases velocity decreases. From table 2, it is found that on increasing Hartmann number strength of the magnetic field decreases. For large Hartmann number, the strength of magnetic fieldi inversely proportional to Hartmann number i.e. independent of the distance from the plates. Figure 1 and 2 show the pattern of the flow of the velocity and distribution of magnetic field respectively.

REFERENCES

[1] M. Jakob and K. Rees, Heat transfer to a fluid in a laminar flow through an annular space, *Trans. AICHE*, **37**(1941), 619-648.

[2] P.A. Mc Cuen, W. M. Kays and W. C. Reynolds, Heat transfer with laminar flow in concentric annuli with constant and variable wall temperature and heat flux. *Report No. AHT-2, Stanford University CA*, 1961.

[3] P. A. Mc Cuen, W. M. Kays and W. C. Renolds, Heat transfer with laminar flow and turbulent flow between parallel planes with constant and variable wall temperature and heat flux, *Report No. AHT-3, Stanford University, CA.* 1962.

[4] H. S. Heaton, W. C. Reynolds and W. M. Kays, Heat transfer with laminar flow in concentric annuli with constant heat flux and simultaneously developing velocity and temperature distributions, *Stanford University, CA.* 1962.

[5] T. Shigenchi and Y. Lee, An analysis on fully developed laminar flow and heat transfer in concentric annuli with moving cores. *Int. J. Heat Mass Transfer.* **34**(1991) 2593- 2601.

[6] P. G. Safiman , On the stability of laminar flow of dusty gas. *J. Fluid Mech.*, **13** (1962), 120-128.

[7] J. Hartman, Hydrodynamics I. Theory of the Laminar Flow of an electrically conductive Liquid in a homogeneous Magnetic Field, *Kgl. Danske Videnskabernes Selskab Math. Fys Med. Copenhagen.* No. **6** ,15 (1937).

[8] J. Hartmann, and F. Lazarus, Hydrodynamics II-Experimental Investigation on the flow of mercury in a Homogeneous Magnetic Field, *Kgl. Dandke Videnskabernes Selskab Math. Flys Med.l Copenhagen* **No. 7 ,14,** (1937).

Jñānābha, Vol. 38, 2008

A NOTE ON THE ERROR BOUND OF A PERIODIC SIGNAL IN HOLDER METRIC BY THE DEFERRED CESARO PROCESSOR

By

Bhavana Soni

Jawaharlal Institute of Technology Borawan (Khargone)-451228, Madhya Pradesh, India

and

Pravin Kumar Mahajan

28, Tilak Path Khargone-451001 Madhya Pradesh, India

E-mail: pravin_mahajan 75@yahoo.com

(Received : November 20, 2007)

ABSTRACT

We determine the error bound of a periodic signal belonging to H_w -space ([6]) by deferred Cesaro-processor ([1]P. 414 and [4], p. 148) and generalize a result of Zygmund ([7], p. 91).

2000 Mathematics Subject Classification : 42A10,42A24.

Keywords: Analog signal, Deferred Cesaro Processor, Modulus of continuity, Holder metric.

1. Definitions and Notations. Lets $(t) \in C^*[0,2\pi]$ be a class of 2π - periodic analog signals and let the Fourier trigonometric series be given by

$$s(t) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) \equiv \sum_{n=0}^{\infty} A_n(t). \qquad \text{...(1.1)}$$

Singh [6] defined the space H_w by

$$H_w = \{s(t) \in C_{2\pi} : |s(t_1) - s(t_2) \le Kw(|t_1 - t_2|)\} \qquad \text{...(1.2)}$$

and the norm $\|\cdot\|_{w^*}$ by

$$\|s\|_{w^*} = \|s\|_c + \sup_{t_1,t_2} \Delta^{w^*} s(t_1,t_2), \qquad \text{...(1.3)}$$

where

$$\|s\|_c = \sup_{0 \le t \le 2\pi} |s(t)|, \qquad \text{...(1.4)}$$

and

$$\Delta^{w^*} s(t_1,t_2) = \sup_{0 \le t \le 2\pi} |s(t)| + \frac{|s(t_1) - s(t_2)|}{w^*(|t_1 - t_2|)}, t_1 \ne t_2 \qquad \text{...(1.5)}$$

and choosing $\Delta^0 s(t_1,t_2)=0, \omega(t)$ and $\omega^*(t)$ being increasing signals of t. if

$$\omega(|t_1-t_2|)\le A|t_1-t_2|^c \qquad ...(1.6)$$

$$\omega^*(|t_1-t_2|)\le K|t_1-t_2|^\beta, 0\le\beta<\alpha\le 1 \qquad ...(1.7)$$

'A' and 'K' being positive constants, then the space

$$H_\alpha=\left\{s(t)\in C_{2\pi}:|s(t_1)-s(t_2)|\le|t_1-t_2|^\alpha, 0<\alpha\le 1\right\}, \qquad ...(1.8)$$

is a Banach spaces (see[5]) and the metric induced by the norm $\| \|_\alpha$ on H_α is said to be a Hölder metric.

Let $s_n(t)$ be the n^{th} parital sums of (1.1) and $Let\{p_n\}$ and $\{q_n\}$ be sequences of non-negative integers satisfying .

$$P_n<q_n \qquad ...(1.9)$$

and $\lim_{n\to\infty} q_n=\infty.$ ...(1.10)

The processor

$$D_n(s_n)=\frac{1}{q_n-p_n}\sum_{k=p_n}^{q_n} s_k(t), \qquad ...(1.11)$$

defines the deferred Cesaro-transform $D(p_n,q_n)$ ([1], see also [4],p.148). It is known [1] that $D(p_n,q_n)$ is regular under conditions (1.9) and (1.10). Note that -D(0,n) is the (C,1) transform and let $\{\lambda_n\}$ be a monotone non- decreasing sequence of positive integers such that $\lambda_1=1$ and $\lambda_{n+1}-\lambda_n\le 1$, then $D(n-\lambda_n,n)$ is same as the n^{th} generalized De la vallee poussian processor [3] generated by the sequence $\{\lambda_n\}$

We shall use following notations:

$$\phi_{t_1}(t)=s(t_1+2t)+s(t_1-2t)-2s(t_1), \qquad ..(1.12)$$

$$K_n(t)=\frac{1}{2(\sin t/2)^2}\left[\sin(p_n+q_n+1)t\sin(q_n+1-p_n)t\right]. \qquad ...(1.13)$$

2. Main Theorem. Using Fejér operator, Zygmund ([7], p.91) has established the following result.

Theorem A. Let $\omega^*(t)$ be a non-negative and increasing signal defined in a right-hand neighbourhood of $t=0$. Suppose that $\omega^*(t)t^{-\alpha}\downarrow$ for $0<\alpha<1$. Let $\omega(t)$ be the

modulus of continuity for a periodic signal $s(t)$, then if

$$\omega(t) = 0\{\omega^*\}(t), \text{as } t \to o^+ \qquad ...(2.1)$$

we have

$$\max_{0 \le t \le 2\pi} |\sigma_n(s;t_1) - s(t_1)| = O\{\omega^*(1/n)\}, \qquad ...(2.2)$$

where $\sigma_n(s;t_1)$ is the Fejer operator.

The object of the present note is to generalize the above result shall establish following:

Theorem. Let $\omega^*(t)$ be a non-negative and increasing signal defined in the right-hand neighbourhood of t=0. Suppose that $\omega^*(t)t^{-\alpha} \downarrow$ for $0 < \alpha < 1$. Let $\omega(t)$ be the modulus of continuity for a periodic signal $s(t)$, then for $s(t) \in H_\omega, 0 \le \beta < \alpha \le 1$ and

$$\omega(t) = o\{\omega^*\}(t), \text{as } t \to o^+. \qquad ...(2.3)$$

We have

$$\|D_n(s_n) - s(t_1)\|_{\omega^*} = O\left[\left\{\log\left(1 + \frac{q_n}{q_n - p_n}\right)\right\}^{\frac{\beta}{\eta}} \left\{\omega^*\left(\frac{1}{q_n - p_n}\right)\right\}^{1-\frac{\beta}{\eta}}\right].$$

3. **Proof of theorem.** Following Zygmund [7], we have

$$D_n(s_n) - s(t_1) = \frac{1}{q_n - p_n} \sum_{k=p_n}^{q_n} s_n(t) - s(t_1)$$

$$= \frac{1}{(q_n - p_n)\pi} \int_0^{\pi/2} \phi(t) K_n(t) dt.$$

We write

$$E_n(t_1) = D_n(s_n) - s(t_1)$$

$$= \frac{1}{(q_n - p_n)\pi} \int_0^{\pi/2} \phi(t) K_n(t) dt$$

and

$$E(t_1, t_2) = |E(t_1) - E(t_2)|$$

$$\le \frac{1}{\pi(q_n - p_n)} \int_0^{\pi/2} |\phi_{t_1}(t) - \phi_{t_2}(t)| K_n(t) dt$$

$$= \frac{1}{\pi(q_n - p_n)}\left(\int_0^{1/(q_n-p_n)} + \int_{1/(q_n-p_n)}^{\pi/2} \right) = I_1 + I_2, say$$

It is easy to prove

$$\left|\phi_{t_1}(t) - \phi_{t_2}(t)\right| \le 4K\omega(|t|) \qquad ...(3.1)$$

and $$\left|\phi_{t_1}(t) - \phi_{t_2}(t)\right| \le 4K\omega(|t_1 - t_2|). \qquad ...(3.2)$$

Now using (3.1), we have

$$|I_1| = O\left(\frac{1}{(q_n - p_n)}\right)\int_0^{1/(q_n-p_n)} \frac{\omega(t)}{\sin^2 t}\sin(p_n + q_n + 1)t \ \sin(q_n + 1 - p_n)t \ dt$$

$$= \left(\frac{1}{(q_n - p_n)}\right)\int_0^{1/(q_n-p_n)} \frac{O\{\omega^*(t)\}}{t^2}(p_n + q_n + 1)t(q_n + 1 - p_n)t \ dt$$

$$= O(1)\frac{p_n + q_n}{q_n - p_n}\omega^*\left(\frac{1}{(q_n - p_n)}\right).$$

Let $\delta = p_n / q_n < 1$

then $$\lim_{n\to\infty}\frac{p_n + q_n}{q_n - p_n} = \frac{1 + \frac{p_n}{q_n}}{1 - \frac{p_n}{q_n}} = \frac{1+\delta}{1-\delta}$$

thus $$|I_1| = O\left\{\omega^*\left(\frac{1}{q_n - p_n}\right)\right\}. \qquad ...(3.3)$$

Again using (3.1) we have

$$|I_1| = O\left(\frac{1}{q_n - p_n}\right)\int_{1/(q_n-p_n)}^{\pi/2} \frac{\omega(t)}{t^2}dt$$

$$= O\left(\frac{1}{q_n - p_n}\right)\int_{1/(q_n-p_n)}^{\pi/2} \frac{o\{\omega^*(t)\}}{t^{\mu}}t^{\mu-2}dt$$

$$= O\left(\frac{1}{q_n - p_n}\right)\frac{\omega^*\left(\frac{1}{q_n - p_n}\right)}{\left(\frac{1}{q_n - p_n}\right)^{\mu}}\left\{(q_n - p_n)^{-\mu+1}\right\}$$

$$= O\left\{\omega^*\left(\frac{1}{q_n - p_n}\right)\right\}. \qquad ...(3.4)$$

Now from (3.2), we have

$$|I_1| = O\left(\frac{1}{q_n - p_n}\right)\int_0^{1/(q_n-p_n)} \left|\phi_{t_1}(t) - \phi_{t_2}(t)\right|\left|k_n(t)\right| dt$$

$$= O\left(\frac{1}{q_n - p_n}\right)\left(\int_0^{1/q_n} + \int_{1/q_n}^{1/q_n - p_n}\right) = I_{11} + I_{12}, (say)$$

$$I_{11} = o\left(\frac{1}{q_n - p_n}\right)\int_0^{1/q_n} \frac{\omega^*(|t_1 - t_2|)}{t^2}(p_n + q_n + 1)(q_n - p_n + 1)t^2 dt$$

$$= O\left(q_n \int_0^{1/q_n} \omega^*(|t_1 - t_2|) dt\right)$$

$$= O\{\omega^*(|t_1 - t_2|)\} \qquad ...(3.5)$$

and

$$I_{12} = O\left(\frac{1}{q_n - p_n}\right)\int_{1/q_n}^{1/(q_n-p_n)} \frac{\omega^*(|t_1 - t_2|)}{\sin^2 t}(\sin(p_n + q_n + 1))(\sin(q_n - p_n + 1))t \; dt$$

$$= O\left(\frac{1}{q_n - p_n}\right)\int_{1/q_n}^{1/q_n - p_n} \frac{\omega^*(|t_1 - t_2|)}{t^2}(q_n - p_n + 1)t \; dt$$

$$= O\left\{\omega^*(|t_1 - t_2|)\log\frac{q_n}{q_n - p_n}\right\}, \qquad ...(3.6)$$

thus

$$I_1 = O\left\{\omega^*(|t_1 - t_2|)\log\left(1 + \frac{q_n}{q_n - p_n}\right)\right\}. \qquad ...(3.7)$$

Again from (3.3)

$$|I_2| = O\left(\frac{1}{q_n - p_n}\right)\int_{1/(q_n-p_n)}^{\pi/2} \frac{\omega^*(|t_1 - t_2|)}{t^2} dt$$

$$= O\{\omega^*(|t_1 - t_2|)\}. \qquad ...(3.8)$$

Now noting that

$$I_r = I_r^{1-\beta/\eta} I_r^{\beta/\eta}, r = 1, 2, \qquad ...(3.9)$$

we have from (3.3) and (3.7)

$$I_1 = O\left[\left\{\omega^*(|t_1 - t_2|)\log\left(1 + \frac{q_n}{q_n - p_n}\right)\right\}^{\frac{\beta}{\eta}}\left\{\omega^*\left(\frac{1}{q_n - p_n}\right)\right\}^{1-\frac{\beta}{\eta}}\right], \qquad ...(3.10)$$

and from (3.4) and (3.8)

$$I_2 = O\left[\{\omega^*(|t_1 - t_2|)\}^{\frac{\beta}{\eta}}\left\{\omega^*\left(\frac{1}{q_n - p_n}\right)\right\}^{1-\beta/\eta}\right]. \qquad ...(3.11)$$

Thus from (3.10) and (3.11), we have

$$\sup_{t_1, t_2}\left|\Delta^{\omega} E_n(t_1, t_2)\right| = \sup\frac{|E_n(t_1) - E_n(t_2)|}{\omega_1^*(|t_1 - t_2|)}$$

$$= O\left[\left\{\log\left(1 + \frac{q_n}{q_n - p_n}\right)\right\}^{\beta/\eta}\left\{\omega^*\left(\frac{1}{q_n - p_n}\right)\right\}^{1-\beta/\eta}\right]. \qquad ...(3.12)$$

It is to be noted that

$$\|E_n(t_1)\|_c = \max_{0 \le t_1 \le 2\pi}|D(s_n) - s|$$

$$= O\left\{\omega^*\left(\frac{1}{q_n - p_n}\right)\right\}. \qquad ...(3.13)$$

Combining (3.12) and (3.13), we get

$$\|D_n(s_n) - s(t_1)\|_{\omega^*} = O\left[\left\{\log\left(1 + \frac{q_n}{q_n - p_n}\right)\right\}^{\frac{\beta}{\eta}}\left\{\omega^*\left(\frac{1}{q_n - p_n}\right)\right\}^{1-\frac{\beta}{\eta}}\right].$$

This completes the proof of theorem 1.

REFERENCES

[1] R.P. Agnew, On deferred Cesàro means, *Ann. Math* .**33**(1932) 413-421.

[2] G. Alexits, *Convergence Problems of Orthogonal Series*, Pergaman Press, 1961.

[3] P. Chandra, Functions of classes L_p and $Lip(\alpha, p)$ generalized De La Valleepussin means, *Math. Student*, **52**(1984), 121-125.

[4] G. Das , T. Ghose and B.K. Ray, Degree of approximation of functions by their Fourier series in the generalized Hölder metric. *Proc. Indian Acad. Sci (Math. Sci)* **106**,no (2) (1996), 139-153.

[5] S. Prossdorff, Zur, Konvergenz der Fourier reihen Hölder stetiger funktionen, *Math. Nachr*, **69**(1975), 7-14.

[6] T. Singh, Degree of approximation of function in a normed spaces, *Publ. Math. Debrecen*, **40**(3-4) (1992), 261-267.

[7] A. Zygmund, *Trigonometric Series*, **Vol. 1**, Cambridge University Press, 1959.

Jñānābha, Vol. 38, 2008

APPLICATIONS OF SYMMETRY GROUPS IN HEAT EQUATION

By
V. G. Gupta
Department of Mathematics
University of Rajasthan Jaipur-302004, Rajasthan
Kapil Pal and Rita Mathur
Department of Mathematics
S. S. Jain Subodh Postgraduate College, Jaipur-302004, Rajasthan
E-mail *guptavguor@rediffmail.com*
(Received : December 10, 2008)

ABSTRACT

The main object of present paper is that to obtain the most general solution for the parital differential equation of one dimensional heat conduction in a finite rod having the thermal diffusivity k_0 using the general prolongation formula for their symmetry.

2000 Mathematics Subject Classification : 17B66, 22Q75, 70G65)
Keywords: Scaling, Translation, Linearity, Galilean-Boost.

1. Introduction.

1.1 The General Prolongation formula.

Let $$v=\sum_{i=1}^{p}\xi^i(x,u)\frac{\partial}{\partial x^i}+\sum_{\alpha=1}^{q}\phi_\alpha(x,u)\frac{\partial}{\partial u^\alpha} \qquad ...(1.1.1)$$

be a vector field defined on an open subset $M\subset X\times U$ where X is the space of independent variables, U is the space of dependent variables, p is the number of independent variables and q is the number of dependent variables for the system. Then n^{th} -prolongation of v is the vector field

$$pr^{(n)}v=v+\sum_{\alpha=1}^{q}\sum_{j}\phi_\alpha^J\left(x,u^{(n)}\right)\frac{\partial}{\partial u_J^\alpha} \qquad ...(1.1.2)$$

defined on the corresponding space $M^{(n)}\subset X\times U^{(n)}$ where X is the space of the independent variables, $U^{(n)}$ is the space of the dependent varivables and the derivatives of the dependent variables up-to n (order of differential equation). The second summation being over all unordered multi-indices $J=(j_1,...,j_k)$ with $1\le j_k\le p, 1\le k\le n$. The coefficient function ϕ_α^J of $pr^{(n)}$v are given by the following formula

$$\phi_\alpha^J(x,u^{(n)}) = D_J\left(\phi_\alpha - \sum_{i=1}^{p}\xi^i u_i^\alpha\right) + \sum_{i=1}^{p}\xi^i u_i^\alpha, i \qquad ...(1.1.3)$$

where $u_i^\alpha = (\partial u^\alpha / \partial x^i)$ and $u_i^\alpha, i = (\partial u_J^\alpha / \partial x^i)$ (see Olver [2], eq. 2.38 and 2.39, p.-110).

1.2 Theorem. Suppose $\Delta_d(x,u^{(n)})$, for $d=1,...,l$ is a system of differential equations of maximal rank defined over $M \subset X \times U$. If G is a local group of transformations acting on M and

$$pr^{(n)}v[\Delta_d(x,u^{(n)})] = 0 \qquad ...(1.2.1)$$

for $d=1,...,l$, whenever $\Delta(x,u^{(n)}) = 0$ for every infinitesimal generator v of G, is a symmetry group of the system, (see Olver[2], eqation 2.25, page 104).

2. Mathematical Analysis.

2.1 The Heat conduction equation:

The one-dimensional conduction of heat in finite rod, without source, with the assumptions

(a)The position of the rod coincides with the x-axis,

(b) the rod is homogeneous,

(c) It is suffciently thin so that the heat is uniformly distributed over its cross section at a given time t,

(d) The surface of the rod is insulated to prevent any loss of heat through the boundary, is governed by parital differential equation in the standard form

$$u_t = k_0 u_{xx} \qquad ...(2.1.1)$$

where $u(x,t)$ is the temperature at the point x at time t and k_0 be the constant thermal diffusivity, which is the second order differential equation with two independent variables and one dependent variable (in our notation given in (1.1) $p=2$, $n=2$ and $q=1$) (see Churchill[1], Simmons[4]).

3 Method. Using the general prolongation formula we obtained the most general solution for one-dimensional heat conduction equation (2.1.1) (see Olver[2] and Olver [3]).

4 Method of Solution.

4.1 Main Theorem. The most general solution of heat conduction equation

$$u_t = k_0 u_{xx} \qquad ...(4.1.1)$$

is given by

$$u = \left(1/\sqrt{1+4\varepsilon_6 t}\right)\exp\left[\varepsilon_3 - \left\{\varepsilon_5 x + \varepsilon_6 x^2 - \varepsilon_5{}^2 t / k_0(1+4\varepsilon_6 t)\right\}\right]$$
$$\times f\left(e^{\varepsilon_4}(x - 2\varepsilon_5 t/1+4\varepsilon_6 t) - \varepsilon_1, e^{-2\varepsilon_4}(t/1+4\varepsilon_6 t) - \varepsilon_2\right) + \alpha(x,t) \qquad ...(4.1.2)$$

where $\varepsilon_1,...,\varepsilon_6$ are real constants and α an arbitrary solution to the heat equation.

By using the following lemmas we proved the main theorem (4.1).

4.2 Lemma:

Let $$v = \xi(x,y,t,u)\frac{\partial}{\partial x} + \tau(x,y,t,u)\frac{\partial}{\partial t} + \phi(x,y,t,u)\frac{\partial}{\partial u} \qquad ...(4.2.1)$$

be a symmery on $X \times U$. Then the smooth coefficient functions ξ, τ and ϕ are given by $\xi = \xi(x,t), \tau = \tau(t)$ and $\phi(x,t,u) = \beta(x,t)u + \alpha(x,t)$ where α and β are arbitary functions.

Proof. Firstly we determine the second prolongation of v by using (1.1.2) to

$$pr^2 v = v + \phi^x \frac{\partial}{\partial u_x} + \phi^t \frac{\partial}{\partial u_t} + \phi^{xx} \frac{\partial}{\partial u_{xx}} + \phi^{xt} \frac{\partial}{\partial u_{xt}} + \phi^{tt} \frac{\partial}{\partial u_{tt}} \qquad ...(4.2.2)$$

and the coefficients present in (4.2.2) can be calculated by using (1.1.3). Using the infinitesimal criterion (1.2.1) takes the form

$$\phi^t = k_0 \phi^{xx} \qquad ...(4.2.3)$$

By substituting the value of ϕ^t and ϕ^{xx} in equation (4.2.3) and replacing u_t by $k_0 u_{xx}$, then equating the coefficient of the terms in the first and second order paritial derivatives of u, finally we find the determining equations as follows

Table1: The Determining Equations Table

Monomial	Cofficient	
$u_x u_{xt}$	$0 = -2k_0 \tau_u$	(a)
u_{xt}	$0 = -2k_0 \tau_x$	(b)
${u_{xx}}^2$	$-{k_0}^2 \tau_u = -{k_0}^2 \tau_u$	(c)
${u_x}^2 u_{xx}$	$0 = -{k_0}^2 \tau_{uu}$	(d)
$u_x u_{xx}$	$-k_0 \xi_u = -2{k_0}^2 \tau_{xu} - 3k_0 \xi_u$	(e)
u_{xx}	$k_0(\phi_u - \tau_t) = -{k_0}^2 \tau_{xx} + k_0(\phi_u - 2\xi_x)$	(f)
${u_x}^3$	$0 = -k_0 \xi_{uu}$	(g)
${u_x}^2$	$0 = k_0(\phi_{uu} - 2\xi_{xu})$	(h)
u_x	$-\xi_t = k_0(2\phi_{xu} - \xi_{xx})$	(j)
	$\phi_t = k_0 \phi_{xx}$	(k) ...(4.2.4)

The reqirement for (a) and (b) is that τ be a function of t.(e) shows that ξ does not depend on u.(h) shows that ϕ be linear in u,so $\phi(x,t,u) = \beta(x,t)u + \alpha(x,t)$ for functions α and β.

4.3 Lemma. The most general infinitesimal symmetry of the Heat conduction

equation has coefficient functions of the form $\xi = c_1 + c_4 x + c_5 2t + c_6 4xt$, $\tau = c_2 + c_4 2t + c_6 4t^2$ and $\phi = \left[c_3 - (1/k_0)\left\{(c_5 x) + c_6\left(x^2 + 2k_0 t\right)\right\}\right] u + \alpha(x,t)$ where $c_1, \ldots, c_6$ are arbitrary constants.

Proof. Using lemma (4.2), the equation (f) require $\tau_t = 2\xi_x$, so $\xi(x,t) = (1/2)\tau_t + \sigma(t)$ where σ is only function of t. We infer from $(j)\xi_t = -2k_0\beta_x$ implies β be at most a quadratic function in x given by $\beta = -(1/8k_0)\tau_{tt}x^2 - (1/2k_0)\sigma_t x + \rho(t)$ where ρ is only function of t. At the end the equation (k) requires that both α and β be the solution of the heat conduction equation, i.e., $\alpha_t = k_0\alpha_{xx}$ and $\beta_t = k_0\beta_{xx}$. Using the determining equation of β, we find that τ is quadratic in t, and σ, ρ is linear in t. Since all the determining equations are satisfied then the most general infinitesimal symmetry of the heat conduction equation has coefficient functions of the form $\xi = c_1 + c_4 x + c_5 2t + c_6 4xt$, $\tau = c_2 + c_4 2t + c_6 4t^2$ and $\phi = \left[c_3 - (1/k_0)\left\{(c_5 x) + c_6\left(x^2 + 2k_0 t\right)\right\}\right] u + \alpha(x,t)$ where $c_1, \ldots, c_6$ are arbitrary constants and $\alpha(x,t)$ is an arbitrary solution of the Heat conduction equation.

4.4 Lemma. The infinitesimal symmetires of the Heat conduction equation is spanned by the six vector field $v_1 = \partial_x, v_2 = \partial_t, v_3 = u\partial_u, v_4 = x\partial_x + 2t\partial_t$, $v_5 = 2t\partial_x + (-1/k_0)xu\partial_u$, $v_6 = 4tx\partial_x + 4t^2\partial_t - \left(\left(x^2 + 2k_0 t\right)/k_0\right)u\partial_u$, and the infinite-dimensional sub algebra $v_\alpha = \alpha(x,t)\partial_u$.

Proof. The proof is evient by using lemma (4.2) and lemma (4.3).

4.5 Lemma. The symmetry Lie algebra $\mathfrak{g}$ of the heat conduction equation is spanned by the six-vector field $v_1, \ldots, v_6$ and the infinite-dimensional sub algebra v_α.

Proof : The commutation relations between these vector fields as follows

Table 2: The Commutation Relation Table

	v_1	v_2	v_3	v_4	v_5	v_6	v_α
v_1	0	0	0	v_1	$(-1/k_0)v_3$	$2v_5$	v_{α_x}
v_2	0	0	0	$2v_2$	$2v_1$	$4v_4 - 2v_3$	v_{α_t}
v_3	0	0	0	0	0	0	$-v_\alpha$
v_4	$-v_1$	$-2v_2$	0	0	v_5	$2v_6$	$v_{\alpha'}$
v_5	$(1/k_0)2v_3$	$-2v_1$	0	$-v_5$	0	0	$v_{\alpha''}$
v_6	$-2v_5$	$2v_3$-$4v_4$	0	$-2v_6$	0	0	$v_{\alpha'''}$
v_α	$-v_{\alpha_x}$	$-v_{\alpha_t}$	v_α	$-v_{\alpha'}$	$-v_{\alpha''}$	$-v_{\alpha'''}$	0

where $\alpha' = x\alpha_x + 2t\alpha_t, \alpha'' = 2t\alpha_x + (x/k_0)\alpha, \alpha''' = 4t(x\alpha_x + t\alpha_t) + ((x^2+2t)/k_0)\alpha$ by the above table 2 we find that ϑ forms the Lie algebra with Lie bracket operation.

5 Result.

5.1 Theorem. The one- parameter groups G_i generated by the v_i *are* given as follows

$$G_1:(x+\varepsilon,t,u), G_2:(x,t+\varepsilon,u), G_3:(x,t,ue^{\varepsilon}), G_4:(xe^{\varepsilon},te^{2\varepsilon},u) G_5:(x+2\varepsilon t,y,t,u.\exp$$

$$(-(\varepsilon x+\varepsilon^2 t)/k_0)), G_\alpha:(x,t+\varepsilon\alpha(x,t)), G_6:(x/(1-4\varepsilon t), t/(1-4\varepsilon t), u.\sqrt{(1-4\varepsilon t)}.\exp(-\varepsilon x^2/k_0(1-4\varepsilon t)))$$

where group $G_i(i=1,\ldots,6,\alpha)$ is a symmetry group.

Proof. The one parameter groups $G_i(i=1,\ldots,6,\alpha)$ are obtained by using the lemma 4.4 and $\exp(\varepsilon v_i)(x,t,u)=(\tilde{x},\tilde{t},\tilde{u})$.

5.2 Theorem. The group invariant solution to the Heat conduction equation corresponding to its different symmetry groups are given by the function

$$u^{(1)} = f(x-\varepsilon,t) u^{(2)} = f(x,t-\varepsilon), u^{(3)} = e^{\varepsilon} f(x,t), u^{(4)} = f(xe^{-\varepsilon}, te^{-2\varepsilon}),$$

$$u^{(5)} \exp((-\varepsilon x+\varepsilon^2 t)/k_0) f(x-2\varepsilon t,t), u^{(\alpha)} = f(x,t)+\varepsilon\alpha(x,t),$$

$$u^{(6)} = \frac{1}{\sqrt{(1+4\varepsilon t)}}.\exp(-\varepsilon x^2/k_0(1+4\varepsilon t)) f(x/(1+4\varepsilon t), t/(1+4\varepsilon t)),$$ where $u=f(x,t)$ be an given solution to the Heat equation, $\alpha(x,t)$ any other solution and ε be a real number.

Proof. The group invariant solutions of Heat conduction equation are obtained by using the relation $(x,t,u)=(\tilde{x},\tilde{t},\tilde{u})$ and putting the values of x,t and u in given solution $u=f(x,t)$ for each symmetry group G_i given in theorem 5.1.

Proof of Main Theorem 4.1. The most general solution $u = gf(x,t)$ of the Heat conduction equation is obtained by group transformations $g=\exp(v_\alpha)$ $\exp(\varepsilon_6 v_6)\ldots\exp(\varepsilon_1 v_1)$ of given solution $u=f(x,t)$ and using theorem 5.2 as follows

$$u = \left(1/\sqrt{(1+4\varepsilon t)}\right)\exp\left[\varepsilon_3 - \left\{\varepsilon_5 x + \varepsilon_6 x^2 - \varepsilon_5{}^2 t/k_0(1+4\varepsilon_6 t)\right\}\right]$$
$$\times\left(e^{-\varepsilon_4}(x-2\varepsilon_5 t/1+4\varepsilon l_6 t)-\varepsilon_1, e^{-2\varepsilon_4}(t/1+4\varepsilon_6 t)-\varepsilon_2\right)+\alpha(x,t)$$

where $\varepsilon_1,\ldots,\varepsilon_6$ are real constant and α an arbitrary solution to the Heat equation.

6. Conclusion: In our investigation the groups G_3 and G_α reflect the linearity of the Heat conduction equation. The G_1 and G_2 are the time and space invariance of the equation respectively, and reflect the fact that Heat conduction equation has constant coefficients. The group G_4 is well known scaling symmetry group. The group G_5 represent a kind of Galiean boost to a moving corrdinate frame. The group G_6 is a genuinely local group of transformations and if $u=c$ be a constant solution then the function

$$u=\left(c/\sqrt{(1+4\varepsilon t)}\right)\exp\left(-\varepsilon x^2/k_0(1+4\varepsilon t)\right)$$

be a solution. The fundamental solution of Heat conduction equation be obtained by substituting $c=\sqrt{\varepsilon/\pi}$, at the point $(x_0,t_0)=(0,(-1/4\varepsilon))$. Now, by translating the above solution in t using G_2, with ε replaced by $-1/4\varepsilon$, we get the fundamental solution of the problem in the form $u=\left(1/\sqrt{1+4\pi t}\right)\exp\left(-x^2/4k_0 t\right)$

7. Discussion. The general solution of Heat equation is invarient under its different symmetry groups $G_i(i=1,\ldots,6,\alpha)$ acting on the independent variables.

8. Special Cases:

8.1 If take $k_0=1$ then the most general solution to the Heat conduction equation

$$u_t=u_{xx} \qquad \ldots(8.1.1)$$

is given

$$u=\left(1/\sqrt{(1+4\varepsilon_6 t)}\right)\exp\left[\varepsilon_3-\left\{\varepsilon_5 x+\varepsilon_6 x^2-\varepsilon_5{}^2 t/(1+4\varepsilon_6 t)\right\}\right]$$
$$\times f\left(e^{-\varepsilon_4}(x-2\varepsilon_5 t/1+4\varepsilon_6 t)-\varepsilon_1, e^{-2\varepsilon_4}(t/1+4\varepsilon_6 t)-\varepsilon_2\right)+\alpha(x,t) \qquad \ldots(8.1.2)$$

where $\varepsilon_1,\ldots,\varepsilon_6$ are real constants and α an arbitrary solution (see Olver[2], page 120).

ACKNOWLEDGEMENT

The authors are grateful to thank of Ex. Professor M.A. Pathan, Department of mathematics, Aligarh Muslim University, Uttar Pradesh, India, for given valuable and useful suggestions towards the improvement of this paper.

REFERENCES

[1] R.V. Churchill, *Operational Mathematics* ,3rd edition, McGraw-Hill Kogakusha Ltd., 1972.

[2] P. J. Olver, *Application of Lie Groups to Differential Equations*, Second Edition, Springer-Verlag-New York, Inc.,1993.

[3] P. J. Olver, Symmetry Groups and Group Invariant Solutions of Parital Differential Equation, *Diff. Geom*, **14** (1979) 497-542.

[4] G. F. Simmons, *Differential Equations with Applications and Historical Notes*, 2nd., Mc. Graw Hill, Inc., New York, 1991.

Jñānābha, Vol. 38, 2008

A GENERALIZATION OF MULTIVARIABLE POLYNOMIALS

By
R.C. Singh Chandel and K.P. Tiwari
Department of Mathematics
D.V. Postgraduate College, Orai-285001, Uttar Pradesh, India
E-mail :rc_chandel@yahoo.com

(Received : October 7, 2007)

ABSTRACT

In the present paper, we introduce multivariable analogue of Chandel polynomials [5] and Chandel-Agrawal polynomials [6], Our polynomials are also generalization of multivariable polynomials due to authors Chandel and Tiwari ([7],(1.5)).

2000 Mathematics subject classification : Primary 33C65; Secondary 33C70.
Keywords: Multivariable Analogue of Chandel and Chandel-Agrawal polynomials, Multivariable polynomials due to Chandel and Tiwari.

1.Introduction. Bell polynomials [8] are defined as

$$(1.1)\quad H_n(g,h)=(-1)^n e^{-hg}D^n e^{hg};\qquad D\equiv\frac{d}{dx},$$

where h is constant and g is some specified function of x.

Shrivastava [9] derived from above the polynomials defined by

$$(1.2)\quad G_n(h,g)=e^{-hg}\left(x\frac{d}{dx}\right)^n e^{hg}.$$

Singh [10] introduced generalized Truesdell polynomials defined by Rodrigues' formula

$$(1.3)\quad T_n^{\alpha}(x,r,p)=x^{-\alpha}e^{px^r}\delta^n\left\{x^{\alpha}e^{-px^r}\right\},\qquad \delta\equiv\frac{d}{dx}.$$

Chandel ([1],[2],[3],[4]) introduced and studied a class of polynomials defined by Rodrigues' formula

$$(1.4)\quad T_n^{(\alpha,k)}(x,r,p)=x^{-\alpha}e^{px^r}\Omega_x^n\left\{x^{\alpha}e^{-px^r}\right\},\quad \Omega_x\equiv x^k\frac{d}{dx},$$

where $k\neq 1$.

For $k\to 1$, (1.4) reduces to (1.3)

Srivastava and Singhal [11] also studied slight variation of (1.4) in the form

(1.5) $G_n^{\alpha}(x,r,p,k)=\frac{1}{n!}x^{-\alpha-kn}\exp(px^r)\left(x^{k+1}\frac{d}{dx}\right)^n\{x^{\alpha}\exp(-px^r)\}$

Further to generalize the polynomials defined by (1.1), (1.2), (1.3) (1.4) and (1.5), Chandel [5] introduced and studied a class of polynomials defined by Rodrigues' formula

(1.6) $G_n(h,g,k)=e^{-hg}\Omega_x^n\{e^{hg}\}$,

where h is independent of x and g is suitable function of x differentiable any number of times.

Later on Chandel and Agrawal [6] to generalize (1.5) introduced a class of polynomials defined by

(1.6) $S_n^{(\alpha,k)}(h,g)=[1-hg(x)]^{\alpha}\Omega_x^n[1-hg(x)]^{-\alpha}$,

where α, h,k are any numbers real or complex independent of x and $g(x)$ is any suitable function of x.

Here in the present paper, we introduce and study multivariable analogue of Chandel polynomials (1.5) and Chandel-Agrawal polynomials (1.6) defined through Rodrigues' formula

(1.7) $G_{n_1,\ldots,n_m}^{(b;h_1,\ldots,h_m;k_1,\ldots,k_m)}(g_1,\ldots,g_m)$

$$=(1+h_1g_1+\ldots+h_mg_m)^{-b}\Omega_{x_1}^{n_1}\ldots\Omega_{x_m}^{n_m}\left\{[1+h_1g_1+\ldots+h_mg_m]^{b}\right\}$$

where b, $h_1,\ldots,h_m$, $k_1,\ldots,k_m$ are real or complex numbers independent of $x_1,\ldots,x_m$; while g_i is function of x_i differetiable any number of times and $\Omega_{x_i}\equiv x_i^{k_i}\frac{\partial}{\partial x_i}$, $k_i\neq 1$; $i=1,\ldots,m$.

It is clear that

(1.8) $\lim_{b\to\infty}G_{n_1,\ldots,n_m}^{(b;h_1/b,\ldots,h_m/b;k_1,\ldots,k_m)}(g_1,\ldots,g_m)$

$$=G_{n_1}(h_1,g_1,k_1)\ldots G_{n_m}(h_m,g_m,k_m).$$

Also in addition for $m=1$, we can write

(1.9) $\lim_{b\to\infty}G_n^{(b,h/b,k)}(g)=G_n(h,g,x)$,

where $G_n(h,g,x)$ are polynomials due to Chandel [5] defined by (1.5)

Also for $m=1$, (1.7) reduces to (1.6). That is

(1.10) $G_n^{(b;h;k)}(g)=S_n^{(-b,k)}(-h,g)$.

where $S_n^{(b,k)}(h,g)$ are polynomials due to Chandel and Agrawal [6] defined through (1.6).

Also chossing $k_i = -1, g_i = \alpha_i \log x_i - p x_i^r$ $(i=1,\ldots,m)$ and replacing b by $-b$, (1.7) reduces to the multivariable polynomials due to authors Chandel and Tiwari ([7],(1.5)] defined by Rodriguies' formula

$$(1.8) \quad T_{n_1,\ldots,n_m}^{(b;\alpha_1;\ldots;\alpha_m;k_1,\ldots,k_m;r_1,\ldots,r_m;p_1,\ldots,p_n)}(x_1,\ldots,x_m)$$

$$= \left[1 - \left(\alpha_1 \log x_1 - p_1 x_1^{r_1}\right) - \ldots - \left(\alpha_m \log x_m - p_m x_m^{r_m}\right)\right]^b$$

$$\Omega_{x_1}^{n_1} \ldots \Omega_{x_m}^{n_m} \left\{ \left(1 - \left(\alpha_1 \log x_1 - p_1 x_1^{r_1}\right) - \left(\alpha_1 \log x_m - p_m x_m^{r_m}\right)\right)^{-b} \right\}$$

where n_i are positive integers , $b, \alpha_i, k_i (\neq 1), r_i, k_i$ are arbitrary numbers real or complex independent of all variables $x_i; i = 1,\ldots.,m$.

2. Generating Relation. Starting with Rodrigues' formula (1.7), we have

$$\sum_{n_1,\ldots,n_m=0}^{\infty} G_{n_1,\ldots,n_m}^{(b;h_1,\ldots,h_m;k_1,\ldots,k_m)}(g_1,\ldots,g_m) \frac{t_1^{n_1}}{n_1!} \cdots \frac{t_m^{n_m}}{n_m!}$$

$$= (1 + h_1 g_1 + \ldots + h_m g_m)^{-b} e^{t_1 \Omega_{x_1} + \ldots + t_m \Omega_{x_m}} \left\{(1 + h_1 g_1 + \ldots + h_m g_m)^b\right\},$$

Thus making an appeal to the well known result due to Chandel ([1, p.105 eq. (2.5)]; also see Srivastava-Singhal [10, p.76 eq. (1.12)])

$$(2.1) \quad e^{t\,\Omega x}\{f(x)\} = f\left(\frac{x}{\left\{1-(k-1)tx^{k-1}\right\}^{\frac{1}{(k-1)}}}\right),$$

where $k \neq 1$ and $f(x)$ admits Taylor's series expansion, we finally derive generating relation

$$(2.2) \quad \sum_{n_1,\ldots,n_m=0}^{\infty} G_{n_1,\ldots,n_m}^{(b;h_1,\ldots,h_m;k_1,\ldots,k_m)}(g_1,\ldots,g_m) \frac{t_1^{n_1}}{n_1!} \cdots \frac{t_m^{n_m}}{n_m!}$$

$$= (1 + h_1 g_1 + \ldots + h_m g_m)^{-b} \left[1 + h_1 g_1 \left[\frac{x}{\left\{1-(k_1-1)t_1 x^{k_1-1}\right\}^{\frac{1}{k_1-1}}}\right] + \right.$$

$$\left. + \ldots + h_m g_m \left[\frac{x}{\left\{1-(k_m-1)t_m x^{k_m-1}\right\}^{\frac{1}{k_m-1}}}\right]\right]^b.$$

3. Applications of Generating Relation. An appeal to generating relation (2.2) gives

$$(3.1)\quad G_{n_1,\ldots,n_m}^{(b+b';h_1,\ldots,h_m;k_1,\ldots,k_m)}(g_1,\ldots,g_m)$$

$$=\sum_{s_1=0}^{n_1}\cdots\sum_{s_m=0}^{n_m}\binom{n_1}{s_1}\cdots\binom{n_m}{s_m}G_{n_1-s_1,\ldots,n_m-s_m}^{(b;h_1,\ldots,h_m;k_1,\ldots,k_m)}(g_1,\ldots,g_m)G_{s_1,\ldots,s_n}^{(b';h_1,\ldots,h_m;k_1,\ldots,k_m)}(g_1,\ldots,g_m),$$

which can be further generalized in the form:

$$(3.2)\quad G_{n_1,\ldots,n_m}^{(b_1+\ldots+b_q;h_1,\ldots,h_m;k_1,\ldots,k_m)}(g_1,\ldots,g_m)$$

$$=\sum_{s_{11}+\ldots+s_{1q}=n_1}\cdots\sum_{s_{m1}+\ldots+s_{mq}=n_m}\prod_{j=1}^{q}\binom{n_1}{s_{ij}}\cdots\binom{n_m}{s_{mj}}G_{s_{11},\ldots,s_{m1}}^{(b_1;h_1,\ldots,h_m;k_1,\ldots,k_m)}(g_1,\ldots,g_m)$$

$$\ldots G_{s_{1q},\ldots,s_{mq}}^{(b_q;h_1,\ldots,h_m;k_1,\ldots,k_m)}(g_1,\ldots,g_m).$$

4. Recurrence Relations. Starting with generating relation (2.2), we have

$$(1+h_1g_1+\ldots+h_mg_m)\sum_{n_1,\ldots,n_m=0}^{\infty}G_{n_1,\ldots,n_m}^{(b+1;h_1,\ldots,h_m;k_1,\ldots,k_m)}(g_1,\ldots,g_m)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_m^{n_m}}{n_m!}$$

$$=\sum_{n_1,\ldots,n_m=0}^{\infty}G_{n_1,\ldots,n_m}^{(b;h_1,\ldots,h_m;k_1,\ldots,k_m)}(g_1,\ldots,g_m)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_m^{n_m}}{n_m!}\left[1+h_1g_1\left[\frac{x_1}{\{1-(k_1-1)t_1x^{k_1-1}\}^{\frac{1}{k_1-1}}}\right]+\right.$$

$$\left.+\ldots+h_mg_m\left[\frac{x_m}{\{1-(k_m-1)t_mx^{k_m-1}\}^{\frac{1}{k_m-1}}}\right]\right]$$

$$=\sum_{n_1,\ldots,n_m=0}^{\infty}G_{n_1,\ldots,n_m}^{(b;h_1,\ldots,h_m;k_1,\ldots,k_m)}(g_1,\ldots,g_m)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_m^{n_m}}{n_m!}+$$

$$\left[\sum_{n_1,\ldots,n_m=0}^{\infty}G_{n_1,\ldots,n_m}^{(b;h_1,\ldots,h_m;k_1,\ldots,k_m)}(g_1,\ldots,g_m)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_m^{n_m}}{n_m!}\right]$$

$$\left[h_1\sum_{s_1=0}^{\infty}\gamma_{s_1}x_1^{s_1}\{1-(k_1-1)t_1x_1^{k_1-1}\}^{-\frac{s_1}{(k_1-1)}}+\ldots+h_m\sum_{s_m=0}^{\infty}\gamma_{s_m}x_m^{s_m}\{1-(k_m-1)t_mx_m^{k_m-1}\}^{-\frac{s_m}{(k_m-1)}}\right]$$

$$=\sum_{n_1,\ldots,n_m=0}^{\infty}G_{n_1,\ldots,n_m}^{(b;h_1,\ldots,h_m;k_1,\ldots,k_m)}(g_1,\ldots,g_m)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_m^{n_m}}{n_m!}$$

$$+\left[\sum_{n_1,\ldots,n_m=0}^{\infty} G_{n_1,\ldots,n_m}^{(b;h_1,\ldots,h_m;k_1,\ldots,k_m)}(g_1,\ldots,g_m)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_m^{n_m}}{n_m!}\right]$$

$$\left[h_1\sum_{s_1=0}^{\infty}\gamma_{s_1}x_1^{s_1}\sum_{r_1=0}^{\infty}\left(\frac{s_1}{k_1-1}\right)_{s_1}\frac{\left[(k_1-1)x_1^{k_1-1}\right]^{r_1}}{r_1!}t_1^{r_1}+\right.$$

$$\left.\ldots+h_m\sum_{s_m=0}^{\infty}\gamma_{s_m}x_m^{s_m}\sum_{r_m=0}^{\infty}\left(\frac{s_m}{k_m-1}\right)_{s_m}\frac{\left[(k_m-1)x_m^{k_m-1}\right]^{r_m}}{r_m!}t_m^{r_m}\right]$$

Now equating the coefficiants of $t_1^{n_1}\ldots t_m^{n_m}$ both the sides, we derive the recurrence relation

(4.1) $(1+h_1g_1+\ldots+h_mg_m)G_{n_1,\ldots,n_m}^{(b+1;h_1,\ldots,h_m;k_1,\ldots,k_m)}(g_1,\ldots,g_m)$

$$=G_{n_1,\ldots,n_m}^{(b;h_1,\ldots,h_m;k_1,\ldots,k_m)}(g_1,\ldots,g_m)+h_1\sum_{s_1=0}^{\infty}\sum_{r_1=0}^{n_1}\gamma_{s_1}x_1^{s_1}\left(\frac{s_1}{k_1-1}\right)\left\{(k_1-1)x_1^{k_1-1}\right\}^{r_1}$$

$$G_{n_1-r_1,n_2,\ldots,n_m}^{(b;h_1,\ldots,h_m;k_1,\ldots,k_m)}(g_1,\ldots,g_m)+\ldots+h_m\sum_{s_m=0}^{\infty}\sum_{r_m=0}^{n_m}\gamma_{s_m}x_m^{s_m}\left(\frac{s_m}{k_m-1}\right)_{r_m}\left\{(k_m-1)x_m^{k_m-1}\right\}^{r_m}$$

$$G_{n_1,\ldots,n_{m-1},n_m-r_m}^{(b;h_1,\ldots,h_m;k_1,\ldots,k_m)}(g_1,\ldots,g_m).$$

5. Differential Recurrence Relations. Differentiating (2.2) partially with respect to t_1, we have

$$\sum_{n_1,\ldots,n_m=0}^{\infty} G_{n_1,\ldots,n_m}^{(b;h_1,\ldots,h_m;k_1,\ldots,k_m)}(g_1,\ldots,g_m)\frac{t_1^{n_1-1}}{(n_1-1)!}\frac{t_2^{n_2}}{n_2!}\cdots\frac{t_m^{n_m}}{n_m!}$$

$$=(1+h_1g_1+\ldots+h_mg_m)^{-b}h_1g'_1x_1^{k_1}\left\{1-(k_1-1)t_1x_1^{k_1-1}\right\}^{-k_1/(k_1-1)}$$

$$b\left[1+h_1g_1\left(\frac{x_1}{\left\{1-(k_1-1)t_1x_1^{k_1-1}\right\}^{1/(k_1-1)}}\right)+\ldots+h_mg_m\left(\frac{x_m}{\left\{1-(k_m-1)t_mx_m^{k_m-1}\right\}^{1/(k_m-1)}}\right)\right]^{b-1}$$

Therefore,

$$(1+h_1g_1+\ldots+h_mg_m)\sum_{n_1,\ldots,n_m=0}^{\infty} G_{n_1,\ldots,n_m}^{(b;h_1,\ldots,h_m;k_1,\ldots,k_m)}(g_1,\ldots,g_m)\frac{t_1^{n_1-1}}{(n_1-1)!}\frac{t_2^{n_2}}{n_2!}\cdots\frac{t_m^{n_m}}{n_m!}$$

$$= bh_1 x_1^{k_1} g'_1 \sum_{s_1=0}^{\infty} \left(\frac{k_1}{k_1-1} \right)_{s_1} \left\{ (k_1-1) x_1^{k_1-1} \right\}^{s_1}$$

$$\sum_{n_1,\ldots,n_m=0}^{\infty} G_{n_1,\ldots,n_m}^{(b-1;h_1,\ldots,h_m;k_1,\ldots,k_m)}(g_1,\ldots,g_m) \frac{t_1^{n_1}}{n_1!} \cdots \frac{t_m^{n_m}}{n_m!}.$$

Thus equating the coefficients of $t_1^{n_1} \ldots t_m^{n_m}$ both the sides, we derive recurrence relation :

(5.1) $$(1 + h_1 g_1 + \ldots + h_m g_m) G_{n_1+1,n_2,\ldots,n_m}^{(b;h_1,\ldots,h_m;k_1,\ldots,k_m)}(g_1,\ldots,g_m)$$

$$= bh_1 x_1^{k_1} g'_1 \sum_{s_1=0}^{n_1} \binom{n_1}{s_1} \left(\frac{k_1}{k_1-1} \right)_{s_1} \left\{ (k_1-1) x_1^{k_1-1} \right\}^{s_1} G_{n_1-s_1,n_2,\ldots,n_m}^{(b-1;h_1,\ldots,h_m;k_1,\ldots,k_m)}(g_1,\ldots,g_m),$$

which suggests that m-recurrence relations can be written in the following unified form :

(5.2) $$(1 + h_1 g_1 + \ldots + h_m g_m) G_{n_1,\ldots,n_{i-1},n_i+1,n_{i+1},\ldots,n_m}^{(b;h_1,\ldots,h_m;k_1,\ldots,k_m)}(g_1,\ldots,g_m)$$

$$= bh_1 x_i^{k_i} g'_i \sum_{s_i=0}^{n_i} \binom{n_i}{s_i} \left(\frac{k_i}{k_i-1} \right)_{s_i} \left\{ (k_i-1) x_i^{k_i-1} \right\}^{s_i} G_{n_1,\ldots,n_{i-1},n_i-s_i,n_{i+1},\ldots,n_m}^{(b-1;h_1,\ldots,h_m;k_1,\ldots,k_m)}(g_1,\ldots,g_m),$$

$$i = 1,\ldots,m.$$

Again differenting (2.2) partially with respect to x_1 we have

$$\sum_{n_1,\ldots,n_m=0}^{\infty} \frac{\partial}{\partial x_1} G_{n_1,\ldots,n_m}^{(b;h_1,\ldots,h_m;k_1,\ldots,k_m)}(g_1,\ldots,g_m) \frac{t_1^{n_1}}{n_1!} \cdots \frac{t_m^{n_m}}{n_m!}$$

$$= -bh_1 g'_1 (1 + h_1 g_1 + \ldots + h_m g_m)^{-b-1} \left[1 + h_1 g_1 \left(\frac{x_1}{\left\{ 1-(k_1-1) t_1 x_1^{k_1-1} \right\}^{1/(k_1-1)}} \right) \right.$$

$$\left. + \ldots + h_m g_m \left(\frac{x_1}{\left\{ 1-(k_m-1) t_m x_m^{k_m-1} \right\}^{1/(k_m-1)}} \right) \right]^b + b(1 + h_1 g_1 + \ldots + h_m g_m)^{-b}$$

$$\left[1 + h_1 g_1 \left(\frac{x_1}{\left\{ 1-(k_1-1) t_1 x_1^{k_1-1} \right\}^{1/(k_1-1)}} \right) + \ldots + h_m g_m \left(\frac{x_m}{\left\{ 1-(k_m-1) t_m x_m^{k_m-1} \right\}^{1/(k_m-1)}} \right) \right]^{b-1}$$

$$h_1 g'_1 \left[\left\{ 1-(k_1-1) t_1 x_1^{k_1-1} \right\}^{-1/(k_1-1)} + (k_1-1) t_1 x_1^{k_1-1} \left\{ 1-(k_1-1) t_1 x_1^{k_1-1} \right\}^{-\left(\frac{k_1+2}{k_1-1} \right)} \right].$$

Therefore,

$$(1+h_1g_1+\ldots+h_mg_m)\sum_{n_1,\ldots,n_m=0}^{\infty}\frac{\partial}{\partial x_1}G_{n_1,\ldots,n_m}^{(b;h_1,\ldots,h_m;k_1,\ldots,k_m)}(g_1,\ldots,g_m)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_m^{n_m}}{n_m!}$$

$$=-bh_1g'_1\sum_{n_1,\ldots,n_m=0}^{\infty}G_{n_1,\ldots,n_m}^{(b;h_1,\ldots,h_m;k_1,\ldots,k_m)}(g_1,\ldots,g_m)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_m^{n_m}}{n_m!}$$

$$+bh_1g'_1\sum_{n_1,\ldots,n_m=0}^{\infty}G_{n_1,\ldots,n_m}^{(b-1;h_1,\ldots,h_m,k_1,\ldots,k_m)}(g_1,\ldots,g_m)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_m^{n_m}}{n_m!}$$

$$\left[\sum_{s_1=0}^{\infty}\left(\frac{1}{k_1-1}\right)_{s_1}\frac{\left\{(k_1-1)x_1^{k_1-1}\right\}^{s_i}}{s_1!}t_1^{s_1}+x_1^{k_1-1}(k_1-1)t_1\sum_{s_1=0}^{\infty}\left(\frac{k_1+2}{k_1-1}\right)_{s_1}\left\{(k_1-1)x_1^{k_1-1}\right\}^{s_1}\frac{t_1^{s_1}}{s_1!}\right].$$

Thus equating the coefficients of $t_1^{n_1}\ldots t_m^{n_m}$, we derive differential recurrence relation.

(5.3) $$(1+h_1g_1+\ldots+h_mg_m)\frac{\partial}{\partial x_1}G_{n_1,\ldots,n_m}^{(b;h_1,\ldots,h_m;k_1,\ldots,k_m)}(g_1,\ldots,g_m)$$

$$=-bh_1g'_1G_{n_1,\ldots,n_m}^{(b-1;h_1,\ldots,h_m;k_1,\ldots,k_m)}(g_1,\ldots,g_m)+bh_1g'_1\sum_{s_1=0}^{n_1}\binom{n_1}{s_1}\left(\frac{1}{k_1-1}\right)_{s_1}\left\{(k_1-1)x_1^{k_1-1}\right\}^{s_i}$$

$$G_{n_1-s_1,n_2,\ldots,n_m}^{(b-1;h_1,\ldots,h_m;k_1,\ldots,k_m)}(g_1,\ldots,g_m)+bh_1g'_1x_1^{k_1-1}(k_1-1)\sum_{s_1=0}^{n_1-1}\binom{n_1-1}{s_1}\left(\frac{k_1+2}{k_1-1}\right)_{s_1}$$

$$\left\{(k_1-1)x_1^{k_1-1}\right\}^{s_1}G_{n_1-s_1,n_2,\ldots,n_m}^{(b-1;h_1,\ldots,h_m;k_1,\ldots,k_m)}(g_1,\ldots,g_m),$$

which further suggests m-differential recurrence relations in the following unified form :

(5.4) $$(1+h_1g_1+\ldots+h_mg_m)\frac{\partial}{\partial x_i}G_{n_1,\ldots,n_m}^{(b;h_1,\ldots,h_m;k_1,\ldots,k_m)}(g_1,\ldots,g_m)$$

$$=-bh_ig'_iG_{n_1,\ldots,n_m}^{(b-1;h_1,\ldots,h_m;k_1,\ldots,k_m)}(g_1,\ldots,g_m)+bh_ig'_i\sum_{s_i=0}^{n_i}\binom{n_i}{s_i}\left(\frac{1}{k_i-1}\right)_{s_i}\left\{(k_i-1)x_i^{k_i-1}\right\}^{s_i}$$

$$G_{n_1,\ldots,n_{i-1},n_i-s_i,n_{i+1},\ldots,n_m}^{(b-1;h_1,\ldots,h_m;k_1,\ldots,k_m)}(g_1,\ldots,g_m)+bh_ig'_ix_i^{k_i-1}(k_i-1)\sum_{s_i=0}^{n_i-1}\binom{n_i-1}{s_i}\left(\frac{k_i+2}{k_i-1}\right)_{s_i}$$

$$\left\{(k_i-1)x_i^{k_i=1}\right\}^{s_i}G_{n_1,\ldots,n_{i-1},n_i-s_i-1,n_{i+1},\ldots,n_m}^{(b-1;h_1,\ldots,h_m;k_1,\ldots,k_m)}(g_1,\ldots,g_m),\qquad i=1,\ldots,m.$$

Now eleminating g'_1 from (5.2) and (5.4), we further finally derive:

$$(5.5)\quad x_i^{k_i}\frac{\partial}{\partial x_i}G_{n_1,\ldots,n_m}^{(b;h_1,\ldots,h_m;k_1,\ldots,k_m)}(g_1,\ldots,g_m)\sum_{s_i=0}^{n_i}\binom{n_i}{s_i}\left(\frac{k_i}{k_i-1}\right)_{s_i}\left\{(k_i-1)x_i^{k_i-1}\right\}^{s_i}$$

$$G_{n_1,\ldots,n_{i-1},n_i-s_i,n_{i+1},\ldots,n_m}^{(b-1;h_1,\ldots,h_m;k_1,\ldots,k_m)}(g_1,\ldots,g_m)$$

$$=G_{n_1,\ldots,n_{i-1},n_i+1,n_{i+1},\ldots,n_m}^{(b;h_1,\ldots,h_m;k_1,\ldots,k_m)}(g_1,\ldots,g_m)\Big[-G_{n_1,\ldots,n_m}^{(b-1;h_1,\ldots,h_m;k_1,\ldots,k_m)}(g_1,\ldots,g_m)$$

$$+\sum_{s_i=0}^{n_i}\binom{n_i}{s_i}\left\{(k_i-1)x_i^{k_i-1}\right\}^{s_i}G_{n_1,\ldots,n_{i-1},n_i-s_i,n_{i+1},\ldots,n_m}^{(b-1;h_1,\ldots,h_m;k_1,\ldots,k_m)}(g_1,\ldots,g_m)$$

$$+x_i^{k_i-1}(k_i-1)\sum_{s_i=0}^{n_i-1}\binom{n_i-1}{s_i}\left(\frac{k_i+2}{k_i-1}\right)_{s_i}\left\{(k_i-1)x_i^{k_i-1}\right\}^{s_i}$$

$$G_{n_1,\ldots,n_{i-1},n_i-s_i-1,n_{i+1},\ldots,n_m}^{(b-1;h_1,\ldots,h_m;k_1,\ldots,k_m)}(g_1,\ldots,g_m)\Big],\qquad i=1,\ldots,m.$$

REFERENCES

[1] R.C.S. Chandel, *Properties of Higher Transcendental Functions and Their Applications*, Ph.D. Thesis, Vikram University, Ujjain, 1970.

[2] R.C.S. Chandel, A new class of polynomials, *Indian J. Math.*, **15** (1973), 41-49.

[3] R.C.S. Chandel, A further note on the polynomials $T_n^{(\alpha,k)}(x,r,p)$, *Indian J. Math.*, **16** (1974), 39-48.

[4] R.C.S. Chandel, Generalized Stirling numbers and polynomials, *Publ. Del. Istitut Matematique*, **22** (36) (1977), 43-48.

[5] R.C.S. Chandel, A further generalization of the class of polynomials $T_n^{(\alpha,k)}(x,r,p)$, *Kyungpook Math. J.*, **14** (1974), 45-54.

[6] R.C.S. Chandel, and S. Agrawal, A generalization of a class of polynomials, *Jñānābha*, **21** (1991) 19-25.

[7] R.C.S. Chandel and K.P. Tiwari, Multivariable analogues of generating Truesdell polynomials, *Jñānābha*, **37**, (2007), 167-176.

[8] J. Riordan, *An Introduction to the Combinatorial Analysis*, 1958.

[9] P.N. Shrivastava, On generalized Stirling numbers and polynomials, *Riv. Mat. Univ. Parma* (2), **11**, (1970), 233-237.

[10] R.P. Singh, On generalized Truesdell polynomails, Riv. Mat. Univ. Parma (2). **8** (1967), 345-353.

[11] H.M. Srivastava and J.P. Singhal, A class of polynomails defined by generalized Rodrigues' formula, *Ann. Mat. Pura Appl.* (4) **90** (1971), 75-85.

Jñānābha, Vol. 38, 2008

THE DISTRIBUTION OF SUM OF MIXED INDEPENDENT RANDOM VARIABLES ONE OF THEM ASSOCIATED WITH $\overline{H}$-FUNCTION

By

Mahesh Kumar Gupta

Department of Mathematics, M.S.J. College,

Bharatpur 321001 , Rajasthan, India

(Received : September 15,2007*)*

ABSTRACT

In the present paper, we shall obtain the distribution of sum of two mixed independent random variables with different probability density functions. One with finite probability density function and the other with infinite probability density function associated with $\overline{H}$-function. The method used is of Laplace transform and its inverse. The result obtained by us is sufficiently genetral in nature due to the presence of the $\overline{H}$-function in the probability density function.

2000 Mathematics Subject Classification : Primary 60Exx, 62Exx; Secondary 62H10.

Keywords: $\overline{H}$-function, Laplace Transform, Distribution Function.

1. Introduciton. In the study of statistical distributions, there is a vast literature in the distribution in the linear combination of several independent random variables when each random variable follows a particular family of distributions. the works of Robins [14], Robins and Pitman [15], Kabe [10], Stacy [20]. Sricastava and Singhal [19], Mathai and Saxena [12], Malik [11], Saxena and Dash [16], Goyal and Agrawal [8], Garg and Gupta [6], Garge [5], Garg and Garg [7], is worth mentioning.

It has been observed that the distribution of sum of several independent random variables when each random variable is of simply infinite or doubly infinite range can easily be calculated by means of characteristic function or moment genetating function. However, when the random variables are distributed over finite renge, these method are not much useful and the power of integral transform method comes sharply into focus.

In this paper, we shall obtain the distribution of two independent random variables, X_1 and X_2, where X_1 possess finite uniform probability density function and X_2 follows infinite probability density function involving $\overline{H}$-function, given by the equations (1.1) and (1.2) respectively. Thus

$$f_1(x_1) = \begin{cases} 1/\alpha & 0 \le x \le a \\ 0 \,, & otherwise \end{cases} \qquad a > 0 \qquad \text{...(1.1)}$$

and

$$f_2(x_2) = \begin{cases} Cx_2^{\lambda-1}e^{-\mu x_2}\,\overline{H}_{P,Q}^{M,N}\left[zx_2^{y}\left| \begin{matrix} (a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q} \end{matrix} \right. \right], x_2 \ge 0 \\ 0, \qquad \text{otherwise} \end{cases} \qquad \text{...(1.2)}$$

where

$$C^{-1} = \mu^{-\lambda}\,\overline{H}_{p+1,Q}^{M,N+1}\left[z\mu^{-\gamma}\left| \begin{matrix} (a_j,\alpha_j;A_j)_{1,N,}(1-\lambda,\gamma;1),(a_j,\alpha_j)_{N+1,Q} \\ (b_j,\beta_j)_{1,M,}(b_j,\beta_j;B_j)_{M+1,Q} \end{matrix} \right. \right] \qquad \text{...(1.3)}$$

and the following conditions are satisfied :

(i) $\gamma > 0, \mu > 0, \lambda + \gamma \min\limits_{1 \le j \le M}(b_j/\beta_j) > 0$,

(ii) $$A = \sum_{j=1}^{M}\beta_j + \sum_{j=1}^{N}A_j\alpha_j - \sum_{j=M+1}^{Q}B_j\beta_j - \sum_{j=N+1}^{P}\alpha_j > 0, \qquad \text{...(1.4)}$$

(iii) The parameters of $\overline{H}$ -function are real and so restricted that $f_2(x_2)$ remains positive for $x_2 \ge 0$.

The $\overline{H}$ function occurring in (1.2) is a generalization of well-known Fox H-function [4]. It has been introduced by Inayat Hussain [9] and represented as follows

$$\overline{H}_{P,Q}^{M,N}[z] = \overline{H}_{P,Q}^{M,N}\left[z\left| \begin{matrix} (a_j,\alpha_j;A_j)_{1,N,}(a_j,\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M,}(b_j,\beta_j;B_j)_{M+1,Q} \end{matrix} \right. \right]$$

$$= \frac{1}{2\pi\omega}\int_{\omega\infty}^{\omega\infty}\bar{\phi}(\xi)z^{\xi}d\xi \;\; \omega = \sqrt{-1}, \qquad \text{...(1.5)}$$

$$\bar{\phi}(\xi) = \frac{\prod\limits_{j=1}^{M}\Gamma(b_j - \beta_j\xi)\prod\limits_{j=1}^{N}\{\Gamma(1-a_j+\alpha_j\xi)\}^{A_j}}{\prod\limits_{j=M+1}^{Q}\{\Gamma(1-b_j+\beta_j\xi)\}^{B_j}\prod\limits_{j=N+1}^{P}\Gamma(a_j-\alpha_j\xi)}, \qquad \text{...(1.6)}$$

where a_j $(j=1,...,P)$ and b_j $(j=1,...,Q)$ are complex parameters, $\alpha_j \geq 0(j=1,...,P)$ and $\beta_j \geq 0(j=1,...,Q)$ (not all zero simultaneously) and the exponents A_j $(j=1,...,N)$ and B_j $(j=M+1,...,Q)$ can take on non-integer values. For the absolute convergence of the $\overline{H}$-function, the sufficient conditions given by (ii) of eq. (1.4) have been given by Buschman and Srivastava [1].

The behaviour of the $\overline{H}$-function for small values of $|z|$ follows easily from a result recently given by Rathie [13,p.306, eq. (6.9)]. We have

$$\overline{H}_{P,Q}^{M,N}[z] = 0\left(|z|^{\alpha}\right)\alpha = \min_{1\leq j\leq M}\left[\mathrm{Re}\left(b_j/\beta_j\right)\right], |z| \to 0. \tag{1.7}$$

When all the exponents A_j and B_j take on integral values, the $\overline{H}$-function reduces to the well-known Fox H-function [4]. We mention below give few some interesting special cases of the $\overline{H}$-function [9, pp.4126-4127], which are not the particular cases of Fox H-function.

(i) The generalized Riemann Zeta function [2, p.27, eq. (1)] and [9, p.4127, eq.(27)]

$$\phi(z,p,\eta) = \sum_{n=0}^{\infty}\frac{1}{(\eta+n)^p}z^n = \overline{H}_{2,2}^{1,2}\left[-z\left|\begin{matrix}(0,1;1),(1-\eta,1;p)\\(0,1),(-\eta,1;p)\end{matrix}\right.\right]. \tag{1.8}$$

The above function is the generalization of the well-known generalized (Hurwitz's) Zeta function $\zeta(p,\eta)$ and Reimann Zeta function $\zeta(p)$ [2,p.24, eq.(1), p.32, eq.(1)].

(ii) The polylogarithm of order p [2, p.30, eq. 1.11 (14)],

$$F(z,p) = \sum_{n=1}^{\infty}\frac{z^n}{n^p} = zf(z,p,1) = -\overline{H}_{2,2}^{1,2}\left[-z\left|\begin{matrix}(1,1;1),(1,1;p)\\(1,1),(0,1;p)\end{matrix}\right.\right]. \tag{1.9}$$

For $p=2$ the above function reduces into Euler's dilogarithm [2,p.31, eq. (22)].

(iii) The exact partition function of the Gaussian model in statistical mechanics [9,p. 4127, eq. 28].

$$\beta' F(d,\varepsilon) = \frac{-(1+\varepsilon)^{-2}}{4\pi^{d/2}} = \frac{-(1+\varepsilon)^{-2}}{4\pi^{d/2}}\overline{H}_{3,2}^{1,3}\left[-(1+\xi)^{-2}\left|\begin{matrix}(0,1;1),(0,1;1),(1/2,1;d)\\(0,1),(-1,1;1+d)\end{matrix}\right.\right]. \tag{1.10}$$

In the $p.d.f$ defined by (1.2), if we reduce $\overline{H}$-function to Fox H-function by taking $A_j = B_j = 1$, we get the $p.d.f.$ defined by Mathai and Saxena [12, eq. (8), p. 163], which on taking the limit as $\mu \to 0$, gives the $p.d.f$ of studied by Srivastava and

Singhal [19]. Further, on specializing the $\overline{H}$ -function as given above we can obtain various new p.d.f. s involving the functions defined by equation (1.8), (1.9) and (1.10).

2. Distribution of the Mixed Independent Random Variables

Theorem. Let X_1 and X_2 be two independent random variables having the probability density function defined by (1.1) and (1.2) respectively. Then the probability density function of

$$Y = X_1 + X_2 \tag{2.1}$$

is given by

$$\begin{aligned} g(y) &= g_1(y), && 0 \le y \le a \\ &= g_1(y) - g_2(y), && a < y < \infty \end{aligned} \tag{2.2}$$

where

$$g_1(y) = \frac{Cy^{-\lambda}}{a} \sum_{n=0}^{\infty} \frac{(-\mu y)^n}{n!} \overline{H}_{p+1,Q+1}^{M,N+1}\left[xy^{\gamma} \middle| \begin{matrix} (a_j,\alpha_j;A_j)_{1,N,} (1-\lambda-n,\gamma;1), (a_j,\alpha_j)_{N+1,p} \\ (b_j\beta_j)_{1,M,} (b_j,\beta_j;B_j)_{M+1,Q}, (-\lambda-n,\gamma;1) \end{matrix} \right]$$

$$y \ge 0 \tag{2.3}$$

and

$$g_2(y) = \frac{C(y-a)^{\lambda}}{a} \sum_{n=0}^{\infty} \frac{\{-\mu(y-a)\}^n}{n!} \overline{H}_{P+1,Q+1}^{M,N+1}\left[z(y-a)^{\gamma} \middle| \begin{matrix} (a_j,\alpha_j;A_j)_{1,N,} (1-\lambda-n,\gamma;1), (a_j,\alpha_j)_{N+1,p} \\ (b_j\beta_j)_{1,M,} (b_j,\beta_j;B_j)_{M+1,Q}, (-\lambda-n,\gamma;1) \end{matrix} \right], y \ge a \tag{2.4}$$

C is given by (1.3) and the following conditions are satisfied :

(i) $\gamma > 0,\ \mu > 0,\ \lambda + \gamma \min\limits_{1 \le j \le M} (b_j/\beta_j) > 0$

(ii) $$A = \sum_{j=1}^{M} \beta_j + \sum_{j=1}^{N} A_j\alpha_j - \sum_{j=M+1}^{Q} B_j\beta_j - \sum_{j=N+1}^{P} \alpha_j > 0 \tag{2.5}$$

(iii) The parameters of $\overline{H}$ - function are real and so restricted that $g_1(y)$, $y \ge 0$ and $g_2(y)$, $y \ge a$, remains positive.

Proof. To obtain the probability density function of $Y = X_1 + X_2$, we use the method of Laplace transform and its inverse. Let the Laplace transform of Y be denoted by

$\phi_y(s)$, then

$$\phi_y(s) = L\{f_1(x_1);s\}L\{f_2(x_2);s\} \tag{2.6}$$

The Laplace transform of $f_1(x_1)$ is a simple integral and to evaluate the Laplace transform of $f_2(x_2)$, we express the $\overline{H}$ -function in terms of Mellin-Barnes type contour integral (1.5), interchange the order of x_2- and ξ-integrals and evaluate x_2 integral as gamma integral to get

$$g(y) = \left(\frac{1-e^{-as}}{s}\right)\frac{C(s+\mu)^{-\lambda}}{a}\overline{H}^{M,N+1}_{p+1,Q}\left[z(s+\mu)^{-\gamma}\middle|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(1-\lambda,\gamma;1),(a_j,\alpha_j)_{N+1,p}\\(b_j\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right] \tag{2.7}$$

Now, we break above expression in two parts, as follows

$$g(y) = \frac{C(s+\mu)^{-\lambda}}{sa}\overline{H}^{M,N+1}_{p+1,Q}\left[z(s+\mu)^{-\gamma}\middle|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(1-\lambda,\gamma;1),(a_j,\alpha_j)_{N+1,p}\\(b_j\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right]$$

$$-\frac{e^{-as}C(s+\mu)^{-\lambda}}{sa}\overline{H}^{M,N+1}_{p+1,Q}\left[z(s+\mu)^{-\gamma}\middle|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(1-\lambda,\gamma;1),(a_j,\alpha_j)_{N+1,p}\\(b_j\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right] \tag{2.8}$$

To obtain the inverse Laplace transform of first term of eq. (2.8), we express the $\overline{H}$ -function in contour integral, collect the terms involving 's' and take its inverse Laplace transform and then use the known result [3, p.238, eq.8]. Writing, the confluent hypergeometric function thus obtained in series form and interpreting the result by definition (1.5), we get the value of $g_1(y)$ as given by the eq. (2.3).

The inverse Laplace transform of second term easily follows by the value of $g_1(y)$ and shifting property for Laplace transform.

3. Special Cases

(i) In the theorem obtained in section 2 if we take $A_i = B_j = 1$, the *p.d.f.* $f_2(x_2)$ reduces to the *p.d.f.* defined by Mathai and Saxena [12] as follows

$$f_2(x_2) = \begin{cases} C_1 x_2^{\lambda-1}e^{-\mu x_2}H^{M,N}_{P,Q}\left[zx_2^{\gamma}\middle|\begin{matrix}(a_j,\alpha_j)_{1,P}\\(b_j,\beta_j)_{1,Q}\end{matrix}\right], x_2 \geq 0 \\ 0, \qquad \text{otherwise}\end{cases} \tag{3.1}$$

where

$$C_1^{-1} = \mu^{-\lambda} H_{P+1,Q}^{M,N+1}\left[z\mu^{-\gamma} \left| \begin{matrix} (1-\lambda,\gamma),(a_j,\alpha_j)_{1,P} \\ (b_j,\beta_j)_{1,Q} \end{matrix} \right. \right]$$

and the corresponding *p.d.f.* of Y as obtained from the eq. (2.2) is given by

$$\begin{aligned} h(y) &= h_1(y), & 0 \le y \le a \\ &= h_1(y) - h_2(y), & a < y < \infty \end{aligned} \tag{3.3}$$

where

$$h_1(y) = \frac{C_1 y^{-\lambda}}{a} \sum_{n=0}^{\infty} \frac{(-\mu y)^n}{n!} H_{P+1,Q+1}^{M,N+1}\left[zy^{\gamma} \left| \begin{matrix} (1-\lambda-n,\gamma),(a_j,\alpha_j)_{1,P} \\ (b_j,\beta_j)_{1,Q},(-\lambda-n,\gamma) \end{matrix} \right. \right], y \ge 0 \tag{3.4}$$

and

$$h_2(y) = \frac{C_1 (y-a)^{\lambda}}{a} \sum_{n=0}^{\infty} \frac{\{-\mu(y-a)\}^n}{n!} H_{P+1,Q+1}^{M,N+1}\left[z(y-a)^{\gamma} \left| \begin{matrix} (1-\lambda-n,\gamma),(a_j,\alpha_j)_{1,P} \\ (b_j,\beta_j)_{1,Q},(-\lambda-n,\gamma) \end{matrix} \right. \right]$$

$$y \ge a \tag{3.5}$$

Further, if we take $a=1$ we get a known result obtained earlier by Garg [5, eq. (2.2), p.79]. This result can be further be reduced to yield the known result recorded in book [17].

(ii) In the Theorem, if we reduce the $\overline{H}$-function to generalized Riemann Zeta function as given by relation (1.8), the *p.d.f.* $f_2(x_2)$ assumes the following form:

$$f_2(x_2) = \begin{cases} C_2 x_2^{\lambda-1} e^{-\mu x_2} \phi(-zx_2^{\gamma}, p, \eta), x_2 \ge 0 \\ 0, \qquad \text{otherwise} \end{cases} \tag{3.6}$$

where

$$C_2^{-1} = \mu^{-\lambda} \overline{H}_{3,2}^{1,3}\left[-z\mu^{-\gamma} \left| \begin{matrix} (0,1;1),(1-\eta,1;p),(1-\lambda,\gamma;1) \\ (0,1),(-\eta,1;p) \end{matrix} \right. \right] \tag{3.7}$$

and the corresponding *p.d.f.* of Y as obtained from the eq. (2.2) is given by

$$\begin{aligned} h(y) &= h_1(y), \ 0 \le y \le a \\ &= h_1(y) - h_2(y), \qquad a < y < \infty \end{aligned} \tag{3.8}$$

where

$$h_1(y)=\frac{C_2y^{-\lambda}}{a}\sum_{n=0}^{\infty}\frac{(-\mu y)^n}{n!}\overline{H}_{3,3}^{1,3}\left[-zy^{\gamma}\middle|\begin{matrix}(0,1;1),(1-\eta,1;p),(1-\lambda-n,\gamma;1)\\(0,1),(-\eta,1;p),(-\lambda-n-\gamma;1)\end{matrix}\right],y\geq 0 \tag{3.9}$$

and

$$h_2(y)=\frac{C_2(y-a)^{-\lambda}}{a}\sum_{n=0}^{\infty}\frac{\{-\mu(y-a)\}^n}{n!}\overline{H}_{3,3}^{1,3}\left[-z(y-a)^{\gamma}\middle|\begin{matrix}(0,1;1),(1-\eta,1;p),(1-\lambda-n,\gamma;1)\\(0,1),(-\eta,1;p),(-\lambda-n,\gamma;1)\end{matrix}\right],y\geq a \tag{3.10}$$

ACKNOWLEDGEMENTS

The author is thankful to University Grants Commission, New Delhi, for providing necessary financial assistance to carry out the present work. The author is also thankful to Dr. Mridula Garg for her valuable suggestions and guidance during the preparation of this paper.

REFERENCES

[1] R.G. Buschman and H.M. Srivastava, The *H*-function associated with a certain class of Feynman integrals, *J. Phys. A : Math. Gen.* **23** (1990), 4707-4710.

[2] A. Erdélyi et al., *Higher Transcendental Functions,* **Vol.1,** McGraw-Hill, New York, 1953.

[3] A. Erdélyi et al., *Table of Integral Transforms,* **Vol. I,** McGraw-Hill, New York, 1954.

[4] C. Fox, The *G* and *H* functions as symmetrical Fourier kernels, *Trans. Amer. Math. Soc.,* **98** (1961) 395-429.

[5] Sheekha Garg, The distribution of the sum of mixed independent random variables, *Ganita Sandesh,* **6(2)** (1992), 78-82.

[6] M. Garg and M.K. Gupta, The distribution of the ratio of two linear functions of independent random variables associated with Fox *H*-function, *Bull. Cal. Math. Soc.,* **88** (1996), 125-130.

[7] M. Garg and Sheekha Garg; The distribution of a polynomials and the ratio of product of rational powers of independent random variables, *Ganita Sandesh,* **9(2)**, (1995), 79-85.

[8] S.P. Goyal and R.K. Agrawal, The distribution of a linear combination and the ratio of products of random variables associated with the multivariable *H*-function, *Jñānābha,* **9/10** (1982).

[9] A.A. Inayat-Hussain, New properties of hypergeometric series derivable from Feynman integrals, II, A generalization of the *H*-function *J. Phys. A: Math. Gen.,* **20** (1987), 4119-4128.

[10] D.G. Kabe, On the exact distribution of a class of multivariable test criteria, *Annals Math. Soc.,* **33** (1962), 1197-1200.

[11] H.J. Malik, Distribution of a linear function and the ratio of two independent linear functions of independent generalized gamma variables, National Research Logistics, **23**, (1976), 339-343.

[12] A.M. Mathai and R.K. Saxena. On the linear combination of stochastic, variables, *Metrika,* **20** (3) (1973), 160-169.

[13] A.K. Rathie, A new generalization of generalized hypergeometric functions, *Le Math. Fasc.* II, **52** (1997), 297-310.

[14] H.E. Robbins. The asymptotic distribution of the sum of random number of variables , *Bull. Amer. Math. Soc.,* **54** (1948) 1151-1161.

[15] H.E. Robbins and J.G. Pitman, Application of the method of mixture to variables, *Australian J. Stat.,* **7** (1949), 110-114.

[16] S.P. Saxena and S.P. Dash, The distribution of linear combinations and ratio of product of

independent variables associated with H-function, *Vijnana Parishad Anusandhan Patrika*, **22** (1979), 57-65.

[17] M.D. Springer. *The Algebra of Random Variables*, John Wiley and Sons, New York (1979).

[18] H.M. Srivastava, K.C. Gupta and S.P. Goyal. The H-Functions of One and Two Variables with Applications, South Asian Publishers, New Delhi, Madras. (1982).

[19] H.M. Srivastava and J.P. Singhal. On a class of generalized hypergeometric distributions, *Jñānābha Sect A*, **2** (1972), 1-9.

[20] E.W. Stacy. A generalization of the gamma distribution, *Anal. Math. Statist.*, **33** (1962), 1187-1192.

Jñānābha, Vol. 38, 2008

ON GENERATING RELATIONSHIPS FOR FOX'S *H*-FUNCTION AND MULTIVARIABLE *H*-FUNCTION

By

B.B. Jaimini and Hemlata Saxena

Government Postgraduate College, Kota-324001, Rajasthan, India

Email : bbjaimini_67@rediffmail.com, E-mail :hemlata_19_75@yahoo.co.in

(Received : January 15, 2008)

ABSTRACT

In this paper we have established some new results on bilinear, bilateral and multilateral generating relationship for Fox's H-function and multivariable H-function. Some known results for the Fox's H-function and multivariable H-function are also obtained as special cases of our main findings.

2000 Mathematics Subject Classification: 33C99, 33C90, 33C45

Keywords : Bilateral generating functions; Fox's H-function; Multivariable H-function; Generating Function Relationships; Combinatorial identities.

1. Introduction and Results Required. Chen and Shrivastava [1] gave a family of linear, bilateral and multilateral generating functions involving the sequence $\left\{\zeta_k^{(\lambda,\rho)}(z)\right\}_{k=0}^{\infty}$ defined by

$$\zeta_k^{(\lambda,\rho)}(z) = {}_uF_{\rho+v}(\alpha_1,\ldots,\alpha_u;\Delta(\rho;1-\lambda-k),\beta_1,\ldots,\beta_v;z) \tag{1.1}$$

where for convenience, $\Delta(\rho;\lambda)$ abbreviates the array of ρ parameters

$$\frac{\lambda}{\rho},\frac{\lambda+1}{\rho},\ldots,\frac{\lambda+\rho-1}{\rho} \qquad (\rho \in N = N_0/\{0\})$$

and for its multivariable extension defined by ([1],p.172, equation (5.21)).

$$Z_k^{\lambda}(\sigma_1,\ldots,\sigma_r;z_1,\ldots,z_r) = \sum_{k_1,\ldots,k_r=0}^{\infty} \frac{A(k_1,\ldots,k_r)}{(1-\lambda-k)_k} z_1^{k_1}\ldots z_r^{k_r} \tag{1.2}$$

$$(K = k_1\sigma_1+\ldots+k_r\sigma_r; k_j \in N_0; \lambda,\sigma_j \in C; j=1,\ldots,r),$$

where $\{A(k_1,\ldots,k_r)\}$ is a suitably bounded multiple sequence of complex numbers and $(\lambda)_k$ denotes the Pochhammer symbol.

$$(\lambda)_k = \frac{\Gamma(\lambda+k)}{\Gamma(\lambda)} = \begin{cases} 1, & (k=0;\lambda\neq 0) \\ \lambda(\lambda+1)\ldots(\lambda+k-1) & (k\in N;\lambda\in C) \end{cases} \tag{1.3}$$

Raina [4] derived the following combinatorial identity as a special case of reduction formula in ([4],p.187, equation (15)).

$$\sum_{k=0}^{\infty}\binom{\lambda+k-1}{k}\binom{\mu+k-1}{k}^{-1}\binom{\alpha+k-1}{k}{}_2F_1\left[\begin{matrix}\lambda+k,\mu-\alpha;\\ \mu+k;\end{matrix}z\right]z^k$$

$$=(1-z)^{-\lambda},\ (|z|<1) \tag{1.4}$$

where $$\binom{n}{k}=\frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n+k+1)}. \tag{1.5}$$

Recently in an earlier paper Jaimini et al. [3] generalized the results of above cited paper [1]. They proved six theorems on the generating function relationship in view of the above result (1.4)

The Fox's H-function defined and represented in the following manner ([2],p.408), see also ([5], p.265, equation (1.1))

$$H_{p,q}^{m,n}\left[z\left|\begin{matrix}\{(a_p,\alpha_p)\}\\ \{(b_q,\beta_q)\}\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_L\phi(\xi)z^{\xi}d\xi \tag{1.6}$$

where

$$\phi(\xi)=\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j\xi)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j\xi)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j\xi)}. \tag{1.7}$$

The multivariable H-function defined and represented in the following manner ([6],pp 251-252, equations (C.1)-(C.3)).

$H[z_1,...,z_r]$

$$=H_{p,q:p_1q_1,...,p_r,q_r}^{o,n:m_1,n_1,...,m_r,n_r}\left[\begin{matrix}z_1\\ \vdots\\ z_r\end{matrix}\left|\begin{matrix}\left(a_j;\alpha_j^{(1)},...,\alpha_j^{(r)}\right)_{1,p}:\left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,p_1};...;\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,p_r}\\ \left(b_j;\beta_j^{(1)},...,\beta_j^{(r)}\right)_{1,q}:\left(d_j^{(1)},\delta_j^{(1)}\right)_{1,q_1};...;\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,q_r}\end{matrix}\right.\right]$$

$$=\frac{1}{(2\pi i)^r}\int_{L_1}\cdots\int_{L_r}\phi_1(s_1)...\phi_r(s_r)\psi(s_1,...,s_r)z_1^{s_1}...z_r^{s_r}ds_1...ds_r; \tag{1.8}$$

where $i=\sqrt{(-1)}$;

$$\phi_i(s_i) = \frac{\prod_{j=1}^{m_i}\Gamma\left(d_j^{(i)} - \delta_j^{(i)}s_i\right)\prod_{j=1}^{n_i}\Gamma\left(1 - c_j^{(i)} + \gamma_j^{(i)}s_i\right)}{\prod_{j=m_i+1}^{q_i}\Gamma\left(1 - d_j^{(i)} + \delta_j^{(i)}s_i\right)\prod_{j=n_i+1}^{p_i}\Gamma\left(c_j^{(i)} - \gamma_j^{(i)}s_i\right)} \quad \forall i \in \{1,\dots,r\} \tag{1.9}$$

$$\psi(s_1,\dots,s_r) = \frac{\prod_{j=1}^{n}\Gamma\left(1 - a_j + \sum_{j=1}^{r}\alpha_j^{(i)}s_i\right)}{\prod_{j=n+1}^{p}\Gamma\left(a_j - \sum_{j=1}^{r}\alpha_j^{(i)}s_i\right)\prod_{j=1}^{q}\Gamma\left(1 - b_j + \sum_{j=1}^{r}\beta_j^{(i)}s_i\right)}. \tag{1.10}$$

In this paper some generating relations for Fox's H-function and multivariable H-function defined in (1.6) and (1.8) respectively are established by following the above cited work of Jaimini et al [3]. The importance of these results lies in the fact that they provide the extensions of the results due to Srivastava and Raina [7] and also provide a wide range of bilinear, bilateral mixed multilateral generating functions for simpler hypergeometric polynomials.

2. Main bilateral generating relationships involving Fox's H-function.

Result-1. Corresponding to an identically nonvanishing function $\Omega_\vartheta(z_1,\dots,z_s)$ of s complex variables $z_1,\dots,z_s (s \in N)$ and of (complex) order ϑ, let

$$\gamma^{(1)}_{m,\vartheta,\rho,\sigma,\lambda}[y,;z_1,\dots,z_s;t] = \sum_{k=0}^{\infty}\frac{a_k\Omega_{\vartheta+\rho k}(z_1,\dots,z_s)t^k}{(mk)!}$$

$$H^{r,s+1}_{u+1}\left[y\left|\begin{matrix}(1-\lambda-mk-\sigma mk,\varepsilon) & \{(c_u,\gamma_u)\} \\ & \{(d_v,\delta_v)\}\end{matrix}\right.\right][a_k \neq 0; k \in N_0; \rho \in N; \vartheta,\sigma \in C] \tag{2.1}$$

and

$$M^{\vartheta,\rho,\lambda,\sigma,\mu,\alpha}_{n,m}[y;z_1,\dots,z_s;\eta] = \sum_{k=0}^{[n/m]}A^{\lambda,\sigma,\mu,\alpha}_{k,n,m}(y;t)$$

$$\binom{\mu+n+\sigma mk-1}{n-mk}^{-1}\binom{\alpha+n+\sigma mk-1}{n-mk}\frac{a_k\Omega_{\vartheta+\rho k}(z_1,\dots,z_s)\eta^k}{(mk)!(n-mk)!} \tag{2.2}$$

where

$$A^{\lambda,\sigma,\mu,\alpha}_{k,n,m}[y;t] = \sum_{l=0}^{\infty}\frac{(\mu-\alpha)_l t^l}{(\mu+n+\sigma mk)_l(l)!}H^{r,s+1}_{u+1,v}\left[y\left|\begin{matrix}(1-\lambda-n-\sigma mk-l,\varepsilon), & \{(c_u,\gamma_u)\} \\ & \{(d_v,\delta_v)\}\end{matrix}\right.\right] \tag{2.3}$$

then

$$\sum_{n=0}^{\infty} M_{n,m}^{\vartheta,\rho,\lambda,\sigma,\mu,\alpha}(y;z_1,\ldots,z_s;\eta)t^n$$

$$=(1-t)^{-\lambda}\gamma_{m,\vartheta,\rho,\sigma,\lambda}^{(l)}\left[\frac{y}{(1-t)^{\varepsilon}};z_1,\ldots,z_s;\frac{\eta t^m}{(1-t)^{(\sigma+1)m}}\right] \tag{2.4}$$

Result-2. Let

$$\gamma_{\vartheta,\rho,m}^{(2)}[y;z_1,\ldots,z_s;t]=\sum_{k=0}^{\infty}\frac{(-1)^{mk}a_k\Omega_{\vartheta+\rho k}(z_1,\ldots,z_s)t^k}{}H_{u,v}^{r,s}\left[y\left|\begin{matrix}\{(c_u,\gamma_u)\}\\\{(d_v,\delta_v)\}\end{matrix}\right.\right] \tag{2.5}$$

and

$$N_{n,m}^{\vartheta,\rho,\lambda,\sigma,\mu,\alpha}[y;z_1,\ldots,z_s;\eta]=\sum_{k=0}^{[n/m]}U_{k,n,m}^{\lambda,\sigma,\mu,\alpha}(y;t)$$

$$\binom{\mu+n+\sigma mk-1}{n-mk}^{-1}\binom{\alpha+n+\sigma mk-1}{n-mk}\frac{(-1)^{mk}a_k\Omega_{\vartheta+\rho k}(z_1,\ldots,z_s)\eta^k}{(n-mk)!} \tag{2.6}$$

where

$$U_{k,n,m}^{\lambda,\sigma,\mu,\alpha}(y;t)=\sum_{l=0}^{\infty}\frac{(\mu-\alpha)_l t^l}{(\mu+n+\sigma mk)_l(l)!}H_{u+1,v+1}^{r,s+1}\left[y\left\{\begin{matrix}(-\lambda-n-\sigma mk-l,\varepsilon),\{(c_u,\gamma_u)\}\\\{(d_v,\delta_v)\},(-\lambda-mk-\sigma mk,\varepsilon)\end{matrix}\right.\right] \tag{2.7}$$

then

$$\sum_{n=0}^{\infty}N_{n,m}^{\ell,\rho,\lambda,\sigma,\mu,\alpha}[y;z_1,\ldots,z_s;\eta]t^n$$

$$=(1-t)^{-(\lambda+1)}\gamma_{\vartheta,\rho,m}^{(2)}\left[\frac{y}{(1-t)^{\varepsilon}};z_1,\ldots,z_s;\frac{\eta t^m}{(1-t)^{(\sigma+1)m}}\right] \quad \text{...(2.8)}$$

Result 3. Let $\gamma_{\vartheta,\rho,m}^{(2)}(y;z_1,..,z_s;t)$ is defined in 2.5 and

$$T_{n,m}^{\vartheta,\rho,\lambda,\sigma,\omega,\mu,\alpha}[y;z_1,\ldots,z_s;\eta]=\sum_{k=0}^{[n/m]}V_{k,n,m}^{\lambda,\sigma,\omega,\mu,\alpha}(y;t)$$

$$\binom{\mu+n+\sigma mk-1}{n-mk}^{-1}\binom{\alpha+n+\sigma mk-1}{n-mk}\frac{(-1)^{mk}a_k\Omega_{\vartheta+\rho_k}(z_1,\ldots,z_s)\eta^k}{(n-mk)!} \quad \text{...(2.9)}$$

where

$$V_{k,n,m}^{\lambda,\sigma,\omega,\mu,\alpha}(y;t)=\sum_{l=0}^{\infty}\frac{(\mu-\alpha)_l t^l}{(\mu+n+\sigma mk)_1 (l)!}$$

$$H_{u+1,v+1}^{r,s+1}\left[y\left|\begin{matrix}(-\lambda-n-\sigma mk-\omega k-l,\varepsilon).\{(C_u,\gamma_u)\}\\ \{(d_v,\delta_v)\},(-\lambda-\sigma mk-(\omega+,)k,\varepsilon)\end{matrix}\right.\right] \quad ...(2.10)$$

then

$$\sum_{n=0}^{\infty}T_{n,m}^{\vartheta,\rho,\lambda,\sigma,\omega,\mu,\alpha}[y;z_1,...,z_s;\eta]t^n$$

$$=(1-t)^{-(\lambda+1)}\gamma_{\partial,\rho,m}^{(2)}\left[\frac{y}{(1-t)^{\varepsilon}};z_1,...,z_s;\frac{\eta t^m}{(1-t)^{(\sigma+1)m+\omega}}\right] \quad ...(2.11)$$

Result -4. Let

$$\gamma_{m,\sigma,\prime,\rho,\omega,\lambda}^{(4)}[y;z_1,...,z_s;t]=\sum_{k=0}^{\infty}(-1)^{mk}a_k\Omega_{\prime+\rho k}(z_1,...,z_s)t^k$$

$$H_{u+1,v+1}^{r,s+1}\left[y\left|\begin{matrix}(1-\lambda-mk-\sigma mk-\omega k,\varepsilon).\{(C_u,\gamma_u)\}\\ \{(d_v,\delta_v)\},(-\lambda-\sigma mk-\omega k-mk,\varepsilon)\end{matrix}\right.\right] \quad ...(2.12)$$

and

$$\theta_{n,m}^{\vartheta,\rho,\lambda,\sigma,\omega,\mu,\alpha}[y;z_1,...,z_s;\eta]=\sum_{k=0}^{[n/m]}W_{k,n,m}^{\lambda,\sigma,\omega,\mu,\alpha}(y;t)$$

$$\binom{\mu+n+\sigma mk-1}{n-mk}^{-1}\binom{\alpha+n+\sigma mk-1}{n-mk}\frac{(-1)^{mk}a_k\Omega_{\vartheta+\rho k}(z_1,...,z_s)\eta^k}{(n-mk)!} \quad ...(2.13)$$

where

$$W_{k,n,m}^{\lambda,\sigma,\omega,\mu,\alpha}(y;t)=\sum_{l=0}^{\infty}\frac{(\mu-\alpha)_l t^l}{(\mu+n+\sigma mk)_l (l)!}$$

$$H_{u+1,v+1}^{r,s+1}\left[y\left|\begin{matrix}(1-\lambda-n-\sigma mk-\omega k-l,\varepsilon).\{(C_u,\gamma_u)\}\\ \{(d_v,\delta_v)\},(-\lambda-\sigma mk-\omega k-mk,\varepsilon)\end{matrix}\right.\right] \quad ...(2.14)$$

then

$$\sum_{n=0}^{\infty}\theta_{n,m}^{\vartheta,\rho,\lambda,\sigma,\omega,\mu,\alpha}\left[y;z_1,\ldots,z_s;\eta\right]=(1-t)^{-\lambda}$$

$$\gamma_{m,\sigma,\vartheta,\rho,\omega,\lambda}^{(4)}\left[\frac{y}{(1-t)^{\varepsilon}};z_1,\ldots,z_s;\frac{\eta t^{m}}{(1-t)^{(\sigma+1)m+\omega}}\right] \qquad \ldots(2.15)$$

Proof of Result-1. We denote the left hand side of the assertion 2.4 of Result-1 by $H[x,y,t]$ then we use the definitions in 2.2 and 2.3, we have:

$$H[x,y,t]=\sum_{n,l=0}^{\infty}\sum_{k=0}^{[n/m]}\frac{(\mu-\alpha)_l t^l}{(\mu+n+\sigma mk)_l(l)!}$$

$$\binom{\mu+n+\sigma mk-1}{n-mk}^{-1}\binom{\alpha+n+\sigma mk-1}{n-mk}\frac{a_k\Omega_{\vartheta+\rho k}(z_1,\ldots,z_s)\eta^k}{(mk)!(n-mk)!}$$

$$H_{u+1,v}^{r,s+1}\left[y\left|\begin{matrix}(1-\lambda-n-\sigma mk-l,\varepsilon),\{(C_u,\gamma_u)\}\\ \{(d_v,\delta_v)\},\end{matrix}\right.\right]t^n$$

Now using the definition of Fox's H-function from 1.6 and changing the order of summation and integration and then on making series rearrangement therein, it takes following form:

$$H[x,y,t]=\frac{1}{2\pi i}\int_L\phi(\xi)y^{\xi}\left[\sum_{n,l=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\mu-\alpha)_l t^l}{(\mu+n+mk+\sigma mk)_l(l)!}\right.$$

$$\left.\binom{\mu+n+mk+\sigma mk-1}{n}^{-1}\binom{\alpha+n+\sigma mk-1}{n}\frac{a_k\Omega_{\vartheta+\rho k}(z_1,\ldots,z_s)}{(mk)!(n)!}\Gamma(\lambda+n+mk+\sigma mk+l+\varepsilon\xi)\eta^k t^{n+mk}\right]d\xi$$

Now in view of the relation

$$\frac{\Gamma(\rho+n+l)}{n!}=(\rho+n)_l\binom{\rho+n-1}{n}\Gamma(\rho) \qquad \ldots(2.16)$$

and then interpreting the inner series into Gauss' hypergeometric function ${}_2F_1$ we have

$$H[x,y,t]=\frac{1}{2\pi i}\int_L\phi(\xi)y^{\xi}\left[\sum_{k=0}^{\infty}\left\{\sum_{n=0}^{\infty}\binom{\mu+n+mk+\sigma mk+\varepsilon\xi-1}{n}\right.\right.$$

$$\binom{\mu+n+mk+\sigma mk-1}{n}^{-1}\binom{\alpha+n+mk+\sigma mk-1}{n}$$

$${}_2F_1\left[\begin{matrix}\lambda+n+mk+\sigma mk+\varepsilon\xi,\mu-\alpha;\\ \mu+n+mk+\sigma mk;\end{matrix}\,t\right]t^n\Bigg\}$$

$$\frac{a_k\Omega_{\vartheta+\rho k}(z_1,\ldots,z_s)\eta^k t^{mk}}{(mk)!}.\Gamma(\lambda+mk+\sigma mk+\varepsilon\xi)\Bigg]d\xi$$

Now using the combinatorial identity 1.4 and then on interpreting the resulting contour into H-function with the help of 1.6, we atonce arrive at the desired result in 2.4

Similarly the proof of Results-2,3,4 would run parallel to that of Result-1, which we have already detailed above fairly adequately.

3. Some Generating Relationships Involving H-Function of Several Variables. The Results-5,6,7,8 given below are established for the multivariable H-function defined in 1.8 by following the corresponding results proved in section-2.

Result-5. Let $\gamma^{(5)}_{\vartheta,\rho,m,\sigma,\lambda}[y_1,\ldots,y_r;z_1,\ldots,z_s;t]=\sum_{k=0}^{\infty}\frac{a_k\Omega_{\vartheta+\rho k}(z_1,\ldots,z_s)t^k}{(mk)!}$

$$H^{0,\upsilon+1:u_1,\upsilon_1;\ldots;u_r,\upsilon_r}_{p+1,q:p_1,q_1;\ldots;p_r,q_r}\left[\begin{matrix}y_1(1-t)^{-\varepsilon_1}\\ \vdots\\ y_r(1-t)^{-\varepsilon_r}\end{matrix}\,\middle|\,(1-\lambda-\sigma mk-mk;\varepsilon_1,\ldots,\varepsilon_r),\right.$$

$$\left.\begin{matrix}\left(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)}\right)_{1,p}:\left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,p_1};\ldots;\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,p_r}\\ \left(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)}\right)_{1,q}:\left(d_j^{(1)},\delta_j^{(1)}\right)_{1,q_1};\ldots;\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,q_r}\end{matrix}\right] \qquad \text{...(3.1)}$$

and

$$R^{\vartheta,\rho,\lambda,\sigma,\mu,\alpha}_{n,m}[y_1,\ldots,y_r;z_1,\ldots,z_s;\eta]=\sum_{k=0}^{[n/m]}B^{\lambda,\sigma,\mu,\alpha}_{k,n,m}[y_1,\ldots,y_r;z_1,\ldots,z_s;t]$$

$$\binom{\mu+n+\sigma mk-1}{n-mk}^{-1}\binom{\alpha+n+\sigma mk-1}{n-mk}\frac{a_k\Omega_{\vartheta+\rho k}(z_1,\ldots,z_s)}{(mk)!(n-mk)!}\eta^k \qquad \text{...(3.2)}$$

where

$$B_{k,n,m}^{\lambda,\sigma,\mu,\alpha}(y_1,\ldots,y_r;t)=\sum_{l=0}^{\infty}\frac{(\mu-\alpha)_l t^l}{(\mu+n+\sigma mk)_l (l)!}$$

$$H_{p+1,q:p_1,q_1;\ldots;p_r,q_r}^{0,\upsilon+1:u_1,\upsilon_1;\ldots;u_r,\upsilon_r}\left[\begin{matrix}y_1\\ \vdots\\ y_r\end{matrix}\middle|(1-\lambda-n-\sigma mk-l;\varepsilon_1,\ldots,\varepsilon_r),\right.$$

$$\left.\begin{matrix}\left(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)}\right)_{1,p}:\left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,p_1};\ldots;\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,p_r}\\ \left(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)}\right)_{1,q}:\left(d_j^{(1)},\delta_j^{(1)}\right)_{1,q_1};\ldots;\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,q_r}\end{matrix}\right] \quad \ldots(3.3)$$

then

$$\sum_{n=0}^{\infty}R_{n,m}^{\vartheta,\rho,\lambda,\sigma,\mu,\alpha}[y_1,\ldots,y_r;y_r;z_1,\ldots,z_s;\eta]t^n$$

$$=(1-t)^{-\lambda}\gamma_{\vartheta,\rho,m,\sigma,\lambda}^{(5)}\left[\frac{y_1}{(1-t)^{\varepsilon_1}},\ldots,\frac{y_r}{(1-t)^{\varepsilon_r}};z_1,\ldots,z_s;\frac{\eta t^m}{(1-t)^{(\sigma+1)m}}\right] \quad \ldots(3.4)$$

Reult-6. Let $\gamma_{\vartheta,m,\rho}^{(6)}[y_1,\ldots,y_r;z_1,\ldots,z_s;t]=\sum_{k=0}^{\infty}(-1)^{mk}a_k\Omega_{\vartheta+\rho k}(z_1,\ldots,z_s)t^k.$

$$H_{p+1,q:p_1,q_1,\ldots,p_r,q_r}^{0,\upsilon+1:u_1,\upsilon_1,\ldots,u_r,\upsilon_r}\left[\begin{matrix}y_1\\ \vdots\\ y_r\end{matrix}\middle|\begin{matrix}\left(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)}\right)_{1,p}:\left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,p_1};\ldots;\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,p_r}\\ \left(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)}\right)_{1,q}:\left(d_j^{(1)},\delta_j^{(1)}\right)_{1,q_1};\ldots;\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,q_r}\end{matrix}\right] \quad \ldots(3.5)$$

and

$$S_{n,m}^{\vartheta,\rho,\lambda,\sigma,\mu,\alpha}[y_1,\ldots,y_r;z_1,\ldots,z_s;\eta]=\sum_{k=0}^{[n/m]}E_{k,n,m}^{\lambda,\sigma,\mu,\alpha}[y_1,\ldots,y_r;z_1,\ldots,z_s;\eta]$$

$$\binom{\mu+n+\sigma mk-1}{n-mk}^{-1}\binom{\alpha+n+\sigma mk-1}{n-mk}\frac{(-1)^{mk}a_k\Omega_{\vartheta+\rho k}(z_1,\ldots,z_s)}{(n-mk)!}\eta^k \quad \ldots(3.6)$$

where

$$E_{k,n,m}^{\lambda,\sigma,\mu,\alpha}(y_1,\ldots,y_r;t)=\sum_{l=0}^{\infty}\frac{(\mu-\alpha)_l t^l}{(\mu+n+\sigma mk)_l (l)!}H_{p+1,q:p_1,q_1;\ldots;p_r,q_r}^{0,\upsilon+1:u_1,\upsilon_1;\ldots;u_r,\upsilon_r}$$

$$\left[\begin{matrix} y_1 \\ \vdots \\ y_r \end{matrix}\middle| \begin{matrix} (-\lambda - n - \sigma mk - l;\varepsilon_1,\ldots,\varepsilon_r)\left(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)}\right)_{1,p} : \left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,p_1};\ldots;\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,p_r} \\ (-\lambda - \sigma mk - mk;\varepsilon_1,\ldots,\varepsilon_r)\left(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)}\right)_{1,q} : \left(d_j^{(1)},\delta_j^{(1)}\right)_{1,q_1};\ldots;\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,q_r} \end{matrix}\right] \quad \ldots(3.7)$$

then

$$\sum_{n=0}^{\infty} S_{n,m}^{\vartheta,\rho,\lambda,\sigma,\mu,\alpha}[y_1,\ldots,y_r;z_1,\ldots,z_s;\eta]t^n$$

$$= (1-t)^{-(\lambda+1)}\gamma_{m,\vartheta,\rho}^{(6)}\left[\frac{y_1}{(1-t)^{\varepsilon_1}},\ldots,\frac{y_r}{(1-t)^{\varepsilon_r}};z_1,\ldots,z_s;\frac{\eta t^m}{(1-t)^{(\sigma+1)m}}\right] \quad \ldots(3.8)$$

Result-7. Let $\gamma_{m,\vartheta,\rho}^{(6)}(y_1,\ldots,y_r;y_r;z_1,\ldots,z_s;t)$ is defined in 3.5

and

$$U_{n,m}^{\vartheta,\rho,\lambda,\sigma,\omega,\mu,\alpha}[y_1,\ldots,y_r;z_1,\ldots,z_s;\eta] = \sum_{k=0}^{[n/m]} F_{k,n,m}^{\lambda,\sigma,\omega,\mu,\alpha}[y_1,\ldots,y_r;t]$$

$$\binom{\mu+n+\sigma mk-1}{n-mk}^{-1}\binom{\alpha+n+\sigma mk-1}{n-mk}\frac{(-1)^{mk}a_k\Omega_{\vartheta+\rho k}(z_1,\ldots,z_s)}{(n-mk)!}\eta^k \quad \ldots(3.9)$$

where

$$F_{k,n,m}^{\lambda,\sigma,\omega,\mu,\alpha}(y_1,\ldots,y_r;t) = \sum_{l=0}^{\infty}\frac{(\mu-\alpha)_l t^l}{(\mu+n+\sigma mk)_l (l)!}\; H_{p+1,q+1:p_1,q_1;\ldots;p_r,q_r}^{0,v+1:u_1,v_1;\ldots;u_r,v_r}$$

$$\left[\begin{matrix} y_1 \\ \vdots \\ y_r \end{matrix}\middle| \begin{matrix} (-\lambda - n - \sigma mk - \omega k - l;\varepsilon_1,\ldots,\varepsilon_r),\left(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)}\right)_{1,p} : \left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,p_1};\ldots;\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,p_r} \\ (-\lambda - \omega k - \sigma mk - mk;\varepsilon_1,\ldots,\varepsilon_r),\left(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)}\right)_{1,q} : \left(d_j^{(1)},\delta_j^{(1)}\right)_{1,q_1};\ldots;\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,q_r} \end{matrix}\right]$$

...(3.10)

then

$$\sum_{n=0}^{\infty} U_{n,m}^{\vartheta,\rho,\lambda,\sigma,\omega,\mu,\alpha}[y_1,\ldots,y_r;z_1,\ldots,z_s;\eta]t^n$$

$$= (1-t)^{-(\lambda+1)}\gamma_{m,\vartheta,\rho}^{(6)}\left[\frac{y_1}{(1-t)^{\varepsilon_1}},\ldots,\frac{y_r}{(1-t)^{\varepsilon_r}};z_1,\ldots,z_s;\frac{\eta t^m}{(1-t)^{(\sigma+1)m+\omega}}\right] \quad \ldots(3.11)$$

Result-8. Let $\gamma^{(8)}_{m,\vartheta,\rho,\lambda,\sigma,\omega}[y_1,...,y_r;z_1,...,z_s;t]=\sum_{k=0}^{\infty}(-1)^{mk}a_k\Omega_{\vartheta+\rho k}(z_1,...,z_s)t^k$.

$$H^{0,\upsilon+1:u_1,\upsilon_1;...;u_r,\upsilon_r}_{p+1,q+1:p_1,q_1;...;p_r,q_r}$$
$$\left[\begin{matrix} y_1\\ \vdots \\ y_r\end{matrix}\middle|\begin{matrix}(1-\lambda-mk-\omega k-\sigma mk;\varepsilon_1,...,\varepsilon_r)\left(a_j;\alpha_j^{(1)},...,\alpha_j^{(r)}\right)_{1,p}:\left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,p_1};...;\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,p_r}\\ (-\lambda-mk-\omega k-\sigma mk;\varepsilon_1,...,\varepsilon_r)\left(b_j;\beta_j^{(1)},...,\beta_j^{(r)}\right)_{1,q}:\left(d_j^{(1)},\delta_j^{(1)}\right)_{1,q_1};...;\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,q_r}\end{matrix}\right] \quad ...(3.12)$$

and

$$V^{\vartheta,\rho,\lambda,\sigma,\omega,\mu,\alpha}_{n,m}[y_1,...,y_r;z_1,...,z_s;\eta]=\sum_{k=0}^{[n/m]}F^{\lambda,\sigma,\omega,\mu,\alpha}_{k,n,m}[y_1,...,y_r;t]$$

$$\binom{\mu+n+\sigma mk-1}{n-mk}^{-1}\binom{\alpha+n+\sigma mk-1}{n-mk}(-1)^{mk}a_k\Omega_{\vartheta+\rho k}(z_1,...,z_s)\eta^k \quad ...(3.13)$$

where

$$G^{\lambda,\sigma,\omega,\mu,\alpha}_{k,n,m}(y_1,...,y_r;t)=\sum_{l=0}^{\infty}\frac{(\mu-\alpha)_l t^l}{(\mu+n+\sigma mk)_l(l)!}$$

$$H^{0,\upsilon+1:u_1,\upsilon_1;...;u_r,\upsilon_r}_{p+1,q+1:p_1,q_1;...;p_r,q_r}$$

$$\left[\begin{matrix} y_1\\ \vdots \\ y_r\end{matrix}\middle|\begin{matrix}(1-\lambda-n-\omega k-\sigma mk-l;\varepsilon_1,...,\varepsilon_r)\left(a_j;\alpha_j^{(1)},...,\alpha_j^{(r)}\right)_{1,p}:\left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,p_1};...;\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,p_r}\\ (-\lambda-\sigma mk-\omega k-mk;\varepsilon_1,...,\varepsilon_r)\left(b_j;\beta_j^{(1)},...,\beta_j^{(r)}\right)_{1,q}:\left(d_j^{(1)},\delta_j^{(1)}\right)_{1,q_1};...;\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,q_r}\end{matrix}\right] (3.14)$$

then

$$\sum_{n=0}^{\infty}V^{\vartheta,\rho,\lambda,\sigma,\omega,\mu,\alpha}_{n,m}[y_1,...,y_r;z_1,...,z_s;\eta]t^n$$

$$=(1-t)^{-\lambda}\gamma^{(6)}_{m,\vartheta,\rho,\lambda,\rho,\omega}\left[\frac{y_1}{(1-t)^{\varepsilon_1}},...,\frac{y_r}{(1-t)^{\varepsilon_r}};z_1,...,z_s;\frac{\eta t^m}{(1-t)^{(\sigma+1)m+\omega}}\right] \quad ...(3.15)$$

4. Special Cases. If in Results-1 to 5 and in Result-8 we take $\Omega_{\vartheta+\rho k}(z_1,...,z_r)\to 1,\sigma=0$ and $\mu=\alpha$ these results reduce to the respective known results in ([7], pp. 37-44, equations (1.10),(1.14),(3.3),(5.3),(6.9),(6.6) at $\beta=0$).

If in the results of section -2, 3 we take $\sigma = 0$, and $\Omega_{\vartheta+\rho k}(z_1,...,z_r) \to 1$ then these results are reduced into certain families of new generating functions associated with the Fox's H-function and multivariable H-function, but we skip the results here.

All the results of section 2 and 3, the product of the essentially arbitary coefficients

$$a_k \neq 0(k \in N_0)$$

and the identically nonvanishing function

$$\Omega_{\vartheta+\rho k}(z_1,...,z_s)(k \in N_0; \rho, s \in N; \vartheta \in C)$$

can indeed be notationally into one set of essentially arbitrary (and indentically nonvanishing) coefficients depending on the order ϑ and on one, two or more variables. In view to applying such results as section 2 above to derive bilateral generating relationships involving Fox's H-function and as section 3 to derive mixed multilateral generating relationship involving multivariable H-function. We find it to be convenient to specialize a_k and $\Omega_k(z_1,...,z_s)$ individually as well as separately. Our general results asserted by section 2 and 3 can be shown to yield various families of bilateral and mixed multilateral generating relation for the specific functions generated in these families but there are not recorded due to lack of space.

REFERENCE

[1] M.P. Chen and H.M. Srivastava, Orthogonality relations and generating functions for Jcobi polynomials and related hypergeometric functions, *Appl. Math. Comput.*, **68** (1995), 153-188.

[2] C. Fox, The *G*-and *H*-functions as symmetrical Fourier kernesls, *Trans. Amer. Math. Soc.*, **98** (1961) 395-429.

[3] B.B. Jaimini, H. Nagar and H.M. Srivastava, Certain classes of generating relations associated with single and multiple hypergeometric functions, *Adv. Stud. Contemp. Math.*, **12** No.1 (2006), 131-142.

[4] R.K. Raina, On a reduction formula involving combinatorial coefficients and hypergeometric functions, *Boll. Un. Math. Ital. A (Ser. 7)* **4** (1990), 183-189.

[5] H.M. Srivastava and R. Panda Some bilateral generating functions for a class of generalized hypergeometric polynomials, *J. Reine Angew., Math.*, **283/284** (1976), 265-274.

[6] H.M. Srivastava, K.C.Gupta and S.P. Goyal, *The H-function of One and Two Variables with Applications* (1982), (New Delhi : South Asian Publishers), pp 251-252.

[7] H.M. Srivastava and R.K. Raina, New generating functions for certain polynomial systems associated with the *H*-functions, *Hokkaido Math. J.*, **10** (1981), 34-45.

Statement of ownership and particulars about the journal

JÑĀNĀBHA

1.	**Place of Publication**	D.V. Postgraduate College Orai-285001, U.P., India
2.	**Periodicity of Publication**	Annual
3.	**Printer's Name** **Nationality** **Address**	Mr. Dheeraj Gupta Indian Customer Gallery, Orai Orai-285001, U.P., India Mobile : 9450296634
4.	**Publisher's Name**	*Dr. R.C. Singh Chandel* For **Vijñāna Parishad of India**
	Nationality **Address**	Indian D.V. Postgraduate College Orai-285001, U.P., India
5.	**Editor's Name** **Nationality** **Address**	Dr. R.C. Singh Chandel Indian D.V. Postgraduate College Orai- 285001, U.P., India
6.	**Name and address of the individuals who own the journal and partners of share holders holding more than one percent of the total capital**	**Vijñāna Parishad of India** Address : D.V. Postgraduate College Orai-285001, U.P., India

I, *Dr. R.C. Singh Chandel* hereby declare that the particulars given above are true to the best of my knowledge and belief.

R.C. Chandel

Dr. R.C. Singh Chandel
Publisher/Editor